CSI 436/536
Introduction to Machine Learning

Review of Linear Algebra and Calculus (2)

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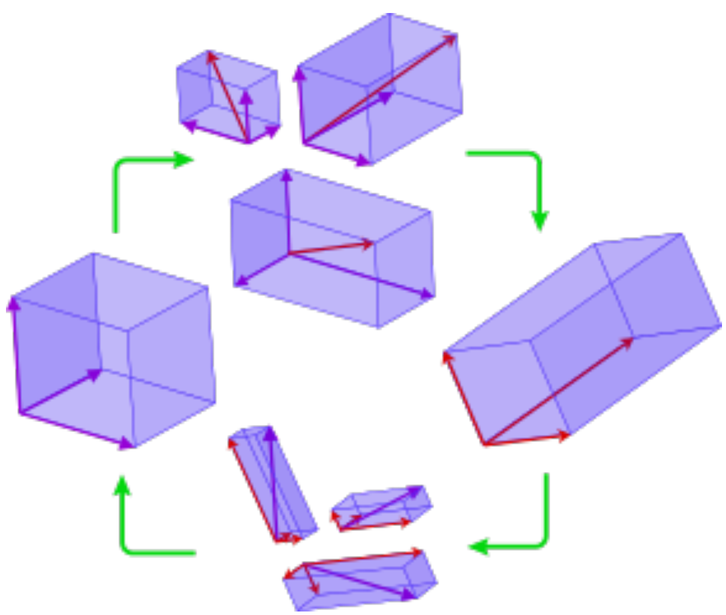
Matrices

- 2D tabular of numbers
- rank-2 tensor
- collection of column vectors, and their space

$$X = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{pmatrix}, \mathrm{col}(X) = \mathrm{span}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n).$$

- collection of row vectors, and their space
- A linear transform

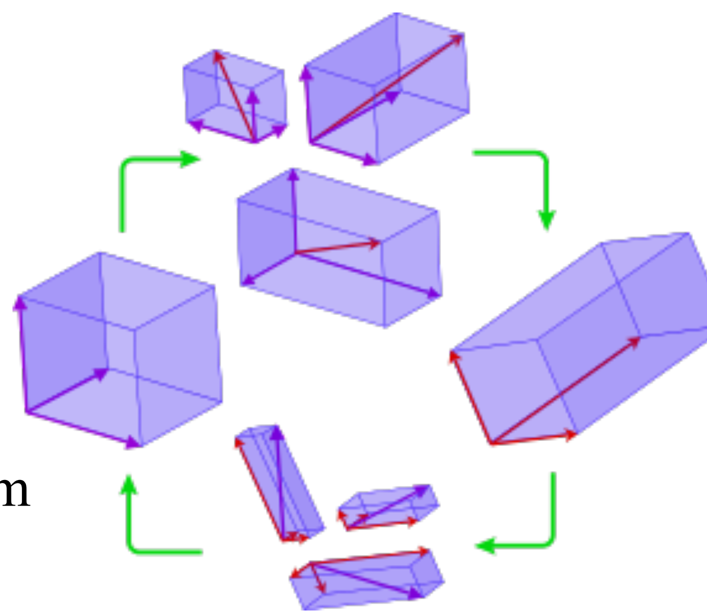


linear transforms and basis

- $\mathrm{T}(\mathbf{v}) = \mathrm{T}(a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n)$
  $= a_1\mathrm{T}(\mathbf{e}_1) + \ldots + a_n\mathrm{T}(\mathbf{e}_n)$
  $= T\mathbf{a}$
- $T\mathbf{a}$ is matrix-vector product
  - T is a matrix each column corresponding to $\mathrm{T}(\mathbf{e}_i)$
  - $\mathbf{a}$ is a vector containing all the values of a_i
- so any linear transform is equivalent to a matrix and vice versa

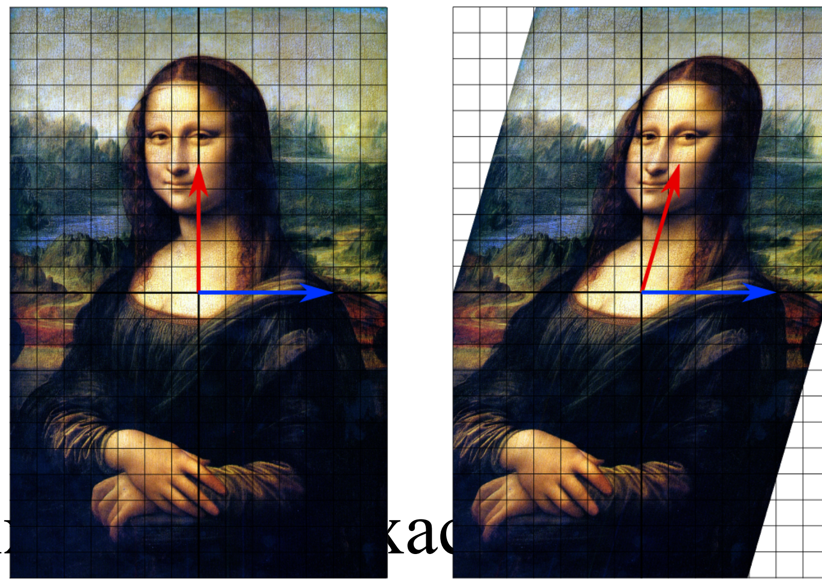
linear transforms

- definition: T is a mapping between vector spaces, and satisfy: $aT(x) + bT(y) = T(ax + by)$
- two equivalent effects
  - move all the points (active transform)
  - move the basis (inactive transform)
  - translation $T(x) = x+c$ is not linear (but can be made so)
- basic “linear transforms
  - rotation
  - scaling
    - isometric scaling
    - anisotropic scaling
  - rotation + scaling = shear transform
  - rotation + isometric scaling = conformal transform
  - rotation + translation = rigid transform
  - rotation + scaling + translation = affine transform



eigenvalue and eigenvector

- An **eigenvector** of a **square** matrix **T** (equivalently a linear transform) is a non-zero complex vector **v** which **T** sends to a complex multiple (the eigenvalue) of itself: **Tv** = λ**v**



- for an n x n matri … xac … alues (counting zero and complex numbers)
  - determinant is a polynomial of n-degree (Caley-Hamilton theorem)

how to solve eigenvalue problem

- solve: $\mathbf{T}\mathbf{v} = \lambda\mathbf{v}$
  - equivalently, we write $(\mathbf{T} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$
  - so that if $(\lambda,\mathbf{v})$ are eigenvalue-eigenvector of $\mathbf{T}$, matrix $(\mathbf{T} - \lambda\mathbf{I})$ is singular
  - or we solve $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$
    - this is known as the characteristic polynomial of matrix T

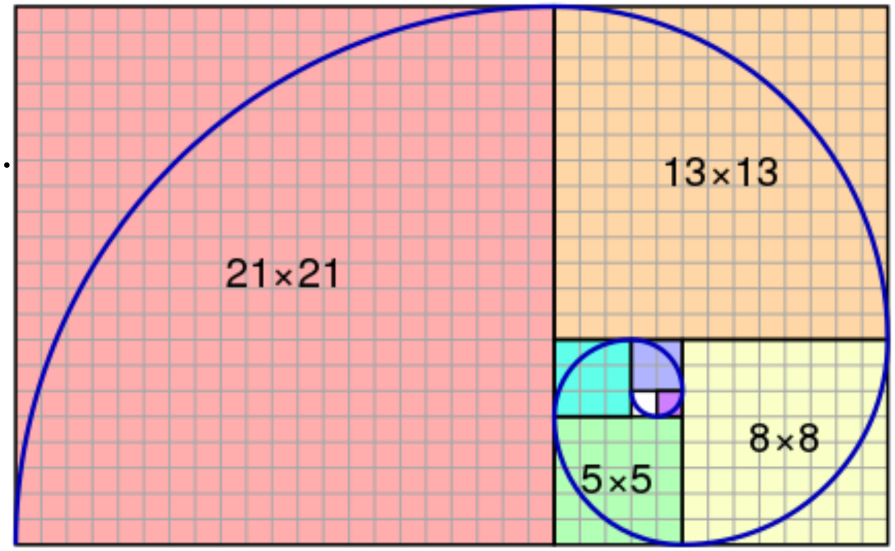
$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \left|\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right| = \lambda^2 + 3\lambda + 2 = 0$$

an application of eigenvalues

Fibonacci number is defined as follows:

$$F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}, n = 3, \cdots$$



$$f_{k+2} = f_{k+1} + f_k$$

$$v_k = \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$$

$$v_{k+1} = \begin{bmatrix} f_{k+2} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} v_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

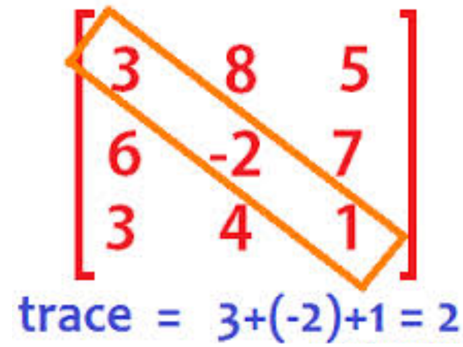
$$\lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$v_{100} = A^{100} v_0$$

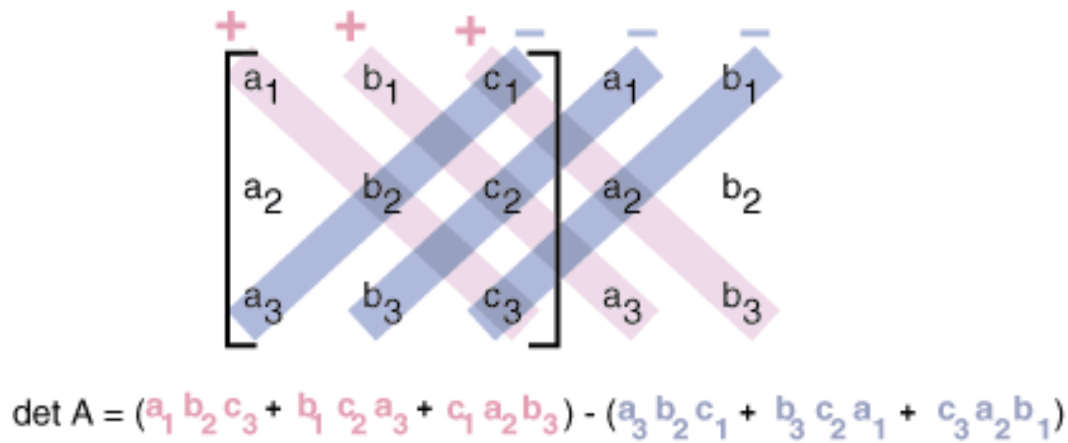
$$A^{100} = S \Lambda^{100} S^{-1}$$

Matrix trace & determinant

- trace = sum of eigenvalues
  - Trace is a linear function of matrix
  - property: $tr(AB) = tr(BA^T)$



- determinant = product of eigenvalues



  - $\det(aA) = a^n \det(A)$, $\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = \det(A)^{-1}$, $\det(e^A) = e^{tr(A)}$
  - If A is not invertible, then det(A) = 0, and vice versa

spectral theorem

- a real symmetric matrix can be decomposed as

$$A = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix} = \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T$$

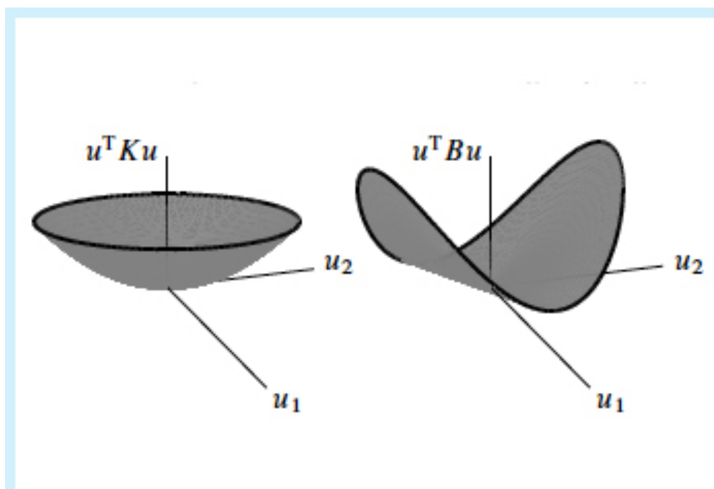
  - $x_1, x_2, \ldots x_n$ are eigenvectors that can be chosen as real vectors
  - $\lambda_1, \lambda_2, \ldots \lambda_n$ are real eigenvalues
- Real symmetric matrix is diagonalizable with orthonormal matrices as $A = U \Lambda U^T$

Some notes

- not every square matrix can be diagonalized, but every square matrix has a Jordan standard form
  - Example: rank-1 matrix uv^T when $u^Tv = 0$
- All normal matrices $A^*A = AA^*$ can be diagonalized
  - symmetric matrices
  - Hermitian matrices
  - Orthogonal matrices
- Every real rectangular matrix can be diagonalized using the singular value decomposition (SVD) as $A = U\Lambda V^T$, where U and V are orthonormal matrices, Λ is a rectangular diagonal matrix with positive diagonals (the singular values)

positive (semi) definite matrices

- $v^T A v$ is the quadratic form of vector v
  - A is p.d. if for any non-zero v, $v^T A v > 0$
    - A has all positive eigenvalues
  - A is p.s.d. if for any non-zero v, $v^T A v \geq 0$
    - A has all nonnegative eigenvalues



equivalence of PD

- a real symmetric matrix is p.d. iff all eigenvalues are positive
  - $\Longrightarrow$: pick any eigenvalue, eigenvector
  - $\Longleftarrow$: use spectral theorem

two important p.s.d. matrices

- for data matrix X (column vectors as data)
  - Gram (inner product) matrix: $G = X^TX$
  - correlation (covariance, outer product) matrix: $C = XX^T$
    - G and C are both positive definite matrices
    - G and C share the same **non-zero** eigenvalues
      - if λ and v are eigenvalue and the corresponding eigenvector of X^TX, we have $X^TXv = \lambda v$
      - then we have $XX^TXv = \lambda Xv$, or $XX^Tu = \lambda Xu$, where
        $u = Xv$
    - G and C's eigenvectors are related by X
    - They are dual to each other