# A comparative study of star graphs and rotator graphs

**Subburajan Ponnuswamy and Vipin Chaudhary**

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Wayne State University

PDCL

**Parallel and Distributed Computing Laboratory**

**Department of Electrical and Computer Engineering**

**Wayne State University, Detroit MI 48202**

**Ph: (313) 577-5802 FAX: (313) 577-1101**

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**Abstract**

Most of the popular interconnection networks can be represented as Cayley graphs. Star graph is one of the extensively studied undirected[1] Cayley graphs, which is considered to be an attractive alternative to the popular binary $n$-cube. The $n$-rotator graph and the cycle prefix digraph are a set of directed Cayley graphs introduced recently. Since the recently introduced directed Cayley graphs have some interesting properties, a comparative study of star and directed Cayley graphs is worthy of study. In this paper we compare the structural and algorithmic aspects of star graphs with that of directed Cayley graphs. In the process we present some new results for star graphs and directed Cayley graphs. We present a formula to calculate the number of nodes at any distance from the identity permutation in star graphs. The minimum bisection width of star and rotator graphs is obtained. Partitioning and fault tolerant parameters for both star and directed Cayley graphs are analyzed. The node disjoint parallel paths and hence the upper bound on the fault diameter of rotator graphs are presented. We compare the minimal path routing in star and rotator graphs using simulation results. Broadcasting and embedding in star and directed Cayley graphs are also compared.

**Key Words:** Bisection width, broadcasting, fault diameter, optimal routing, sphere of locality.

## I. Introduction

The performance of any multiprocessor system depends mainly on the communication efficiency of the underlying interconnection topology. Numerous interconnection networks for both general purpose and special purpose applications have been introduced in the literature. Continuing search for communication efficient symmetric interconnection structures for multiprocessor networks has led to Cayley graphs as possible interconnection networks. Some examples include star graph [1, 2], rotator graph [3], and cycle prefix digraphs [4]. It should be noted that the well studied binary $n$-cube and the cube connected cycles can also be represented as Cayley graphs. Due to their simplicity and the fact that the real world communication links (*e.g.* optical links) are often realized by directed communication links, the directed counterparts of some of the undirected networks have also appeared in the literature, *e.g.*, uni-directional hypercube [5], and uni-directional star

[1] By definition Cayley graphs are directed graphs. When the generators are closed under inverse, as in the case of star graphs, the Cayley graph can be viewed as an undirected graph.

graphs [6]. The $n$-rotator graph and the cycle prefix digraphs are directed by definition. Since these directed Cayley graphs have some interesting properties like easy routing, low diameter and average diameter compared to the star graph, a comparative analysis of these graphs is worthy of study. Throughout this paper, the term *directed Cayley graph* refers to the rotator and cycle prefix digraphs.

The binary $n$-cube is a widely used interconnection topology in practical parallel computers. However, the number of links in the binary $n$-cube, the degree, and the diameter are higher than the star and directed Cayley graphs for the same number of nodes. A comparative study of star and hypercube can be found in [7]. Star graph is one of the extensively studied interconnection networks after binary $n$-cubes. In this paper, we present some new results for star and directed Cayley graphs. We present a formula for calculating the number of nodes at any distance from the identity permutation, using the cyclic structure of the star graph. The minimum bisection width, an important measure in VLSI models, is obtained for both star and directed Cayley graphs. We show that the rotator graphs of size $n!$ have $n-1$ node disjoint parallel paths of length at most $n+1$ for $n > 2$. Other fault tolerant measures like fault diameter, the number of fault free subnetworks available in the presence of faulty nodes, incomplete networks, etc., are discussed. Latency and average diameter are very important in systems with fine grain parallelism and real time computations. We compare the latency and link utilization in the minimal path routing of star and directed Cayley graphs using simulation results.

This paper is organized as follows. In section two, we present the definitions of the networks and other parameters, which will be used in later sections. A method of calculating the number of nodes at any distance less than or equal to the diameter from the identity node in star graph is given in section three. The minimum bisection width of star and directed Cayley graphs are also discussed in section three. Section four is devoted to the partitioning and fault tolerant properties of star and directed Cayley graphs. Node disjoint parallel paths in directed Cayley graphs, fault diameter, and incomplete networks are discussed in this section. Section five presents a comparative analysis of routing, broadcasting, and embedding in star and directed Cayley graphs. Finally section six concludes with a summary of the results.

## II. Preliminaries

We use the terms vertex, node, and permutation interchangeably throughout this paper. The definition of Cayley graph and other group theoretic terms can be found in [1, 2]. We denote the $n$-star graph by $\mathcal{S}_n$. The rotator graph and the cycle prefix digraphs are denoted by $\mathcal{R}_n$ and $\mathcal{C}_n$ respectively. The notation $\mathcal{D}_n$ is used to denote both $\mathcal{R}_n$ and $\mathcal{C}_n$. The general notation $\mathcal{G}_n$ is used to denote all the three graphs discussed in this paper ($\mathcal{S}_n$, $\mathcal{C}_n$ and $\mathcal{R}_n$). The graph $\mathcal{G}_n$ has a set of $(n-1)$ generators $g = \{g_2, g_3, \ldots, g_n\}$. We use $D(\mathcal{G}_n)$ to denote the diameter of $\mathcal{G}_n$. If the node $y$ is adjacent to node $x$ and node $y$ is obtained by applying the generator $g_i$, where $2 \leq i \leq n$. The link connecting $x$ and $y$ is the $i^{th}$ dimension link. A permutation is denoted as $\pi$, and the symbols in $\pi$ are referred to as $\pi[i]$, where $1 \leq i \leq n$.

**Definition 1** *The generators* $\mathcal{S}_n$ *are of the form* $x_1x_2\ldots x_ix_{i+1}\ldots x_n \overset{g_i}{\Rightarrow} x_ix_2x_3\ldots x_1x_{i+1}\ldots x_n$. *The action of the generator* $g_i$ *is the swapping of the first symbol* $x_1$ *with the* $i^{th}$ *symbol, where* $2 \le i \le n$.

**Definition 2** *The generators of the rotator graph* $\mathcal{R}_{(n,k)}$ *are of the form*

$$x_1x_2\ldots x_ix_{i+1}\ldots x_k \Rightarrow \begin{cases} x_2x_3\ldots x_ix_1x_{i+1}\ldots x_k & \text{if } 2 \le i \le k < n \\ x_2x_3\ldots x_ix_{i+1}\ldots x_kx_j & \text{if } k < j \le n \end{cases}$$

*The total number of nodes is* $n!/(n-k)!$, *and the degree of* $\mathcal{R}_{(n,k)}$ *is* $(n-1)$.

**Definition 3** *The generators of the cycle prefix digraph* $\mathcal{C}_{(n,k)}$ *are of the form*

$$x_1x_2\ldots x_ix_{i+1}\ldots x_k \Rightarrow \begin{cases} x_ix_1\ldots x_{i-1}x_{i+1}\ldots x_k & \text{if } 2 \le i \le k < n \\ x_jx_1x_2\ldots x_ix_{i+1}\ldots x_{k-1} & \text{if } k < j \le n \end{cases}$$

*The number of nodes and degree of* $\mathcal{C}_{(n,k)}$ *are the same as that of* $\mathcal{R}_{(n,k)}$.

In this paper we consider the special cases of $\mathcal{R}_{(n,k)}$ and $\mathcal{C}_{(n,k)}$, the $n$-rotator $\mathcal{R}_n$ and the $n$-cycle prefix digraph $\mathcal{C}_n$. The generators of $\mathcal{R}_n$ and $\mathcal{C}_n$ are of the form $x_1x_2\ldots x_ix_{i+1}\ldots x_n \overset{g_i}{\Rightarrow} x_2x_3\ldots x_ix_1x_{i+1}\ldots x_n$ and $x_1x_2\ldots x_ix_{i+1}\ldots x_n \overset{g_i}{\Rightarrow} x_ix_1x_2\ldots x_{i-1}x_{i+1}\ldots x_n$, respectively. Both the set of graphs have $n!$ vertices, degree $(n-1)$, connectivity $(n-1)$, and diameter $(n-1)$. The results obtained for $\mathcal{R}_n$ and $\mathcal{C}_n$ can be easily extended to $\mathcal{R}_{(n,k)}$ and $\mathcal{C}_{(n,k)}$. Most of the results presented here for directed Cayley graphs refer to the rotator graph. We use the terms rotation and generator interchangeably, while discussing rotator graphs. It should be noted that the rotator graphs are isomorphic to the cycle prefix digraphs with the direction of the edges reversed [3]. Therefore, all the results obtained for rotator graphs apply to the cycle prefix digraphs, and vice versa. The $\mathcal{S}_n$ and $\mathcal{R}_n$ are illustrated in Fig. 1 and Fig. 2. In Fig. 1 all the links drawn are undirected links (two unidirectional links in opposite directions). Links denoted by the same alphabet are connected together. In Fig. 2, the directed links are denoted by arrows, and the bold lines without arrows are undirected links. Every node in $\mathcal{R}_n$ has only one undirected link (i.e. the generator $g_2$). The nodes marked with alphabets in Fig. 2 are directed links. The link denoted by an alphabet and $(o)$ denotes that the link is an outgoing link. This link is connected to the node with a link marked with the same alphabet and $(i)$ (means incoming).

**Definition 4** *A container is a set of node-disjoint paths between any two vertices of a graph. The width of a container is the number of node-disjoint paths it includes. The length of a container is the length of the longest path in the container [8].*

Wide containers can be used to send multiple messages from one node to another node in many applications [8]. The length of the container of any graph is the upper bound for the *fault diameter* of the graph [9, 10].

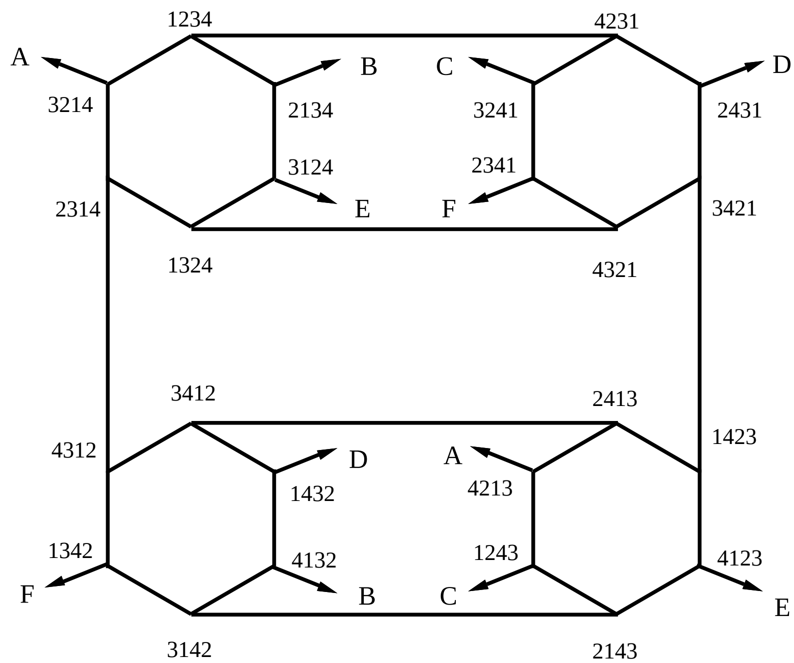



Figure 1: $\mathcal{S}_4$

**Definition 5** *The fault diameter $D_f(G)$ of any graph $G$ with connectivity $k$ is defined as the maximum diameter obtained from $G$ by removing $(k-1)$ nodes [10, 9].*

**Definition 6** *The routing tree of $\mathcal{D}_n$ is the tree structure obtained from the (minimal) paths followed by a message when it is routed from every node to the identity node.*

**Definition 7** *Any permutation $\pi = x_1x_2x_3\ldots x_ix_{i+1}\ldots x_n$ is divided into two regions, a leading unsorted region $x_1x_2x_3\ldots x_i$ and a trailing sorted region $x_{i+1}\ldots x_n$, where $x_{i+1} < x_{i+2} < \ldots < x_{n-1} < x_n$, $x_i > x_{i+1}$, and $(n-i)$ is the length of the sorted sequence.*

**Definition 8** *Two nodes $x$ and $y$ are said to be at bi-distance $d_b$, where $d_b$ is the maximum of the distance from $x$ to $y$ and from $y$ to $x$.*

Since all the links are undirected in $\mathcal{S}_n$, the distance from $x$ to $y$ is the same as the distance from $y$ to $x$. However, *bi-distance* can be used as a measure, for the number of nodes to which a node can communicate in both the directions in $\mathcal{D}_n$.

**Definition 9** *The bisection width B(G) of a graph G is defined as the number of channels that have to be cut in order to bisect the network G into two equal halves.*

## III. Sphere of Locality and Bisection Width

### A. *Number of Nodes*

It is known that the calculation of the number of nodes at any distance from the identity node in $\mathcal{S}_n$ is not trivial. In this section we analyze the number of nodes at any given distance in $\mathcal{S}_n$ and

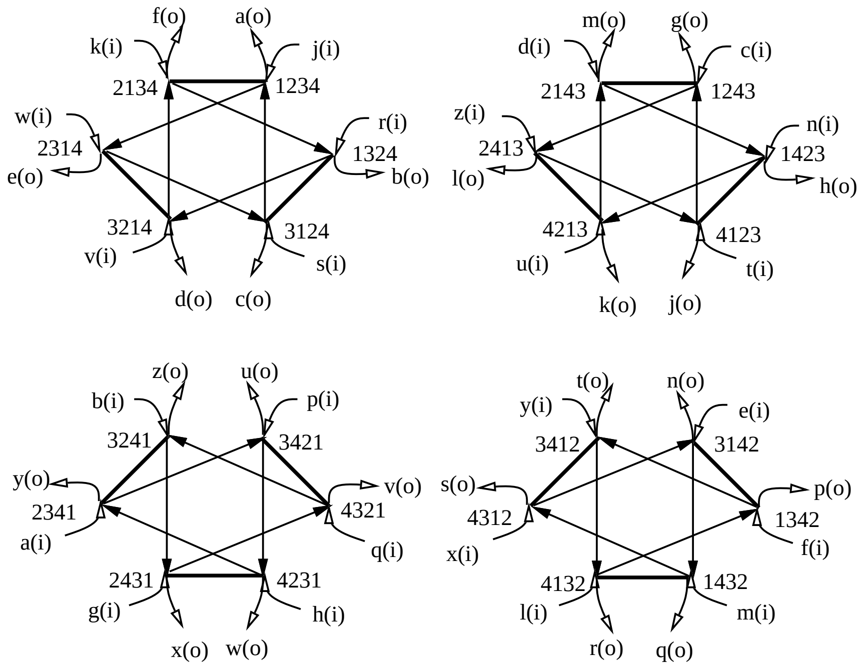



Figure 2: $\mathcal{R}_4$

$\mathcal{D}_n$. We present a formula for calculating the number of nodes at any distance from the identity permutation in $\mathcal{S}_n$. The concentration of nodes at various distances from the identity node is analyzed.

**Lemma 1** *The number of permutations of $n$ symbols with $c$ cycles of length greater than two is*

$$E^c(n) = \sum_{k=0}^{c} \left((-1)^k \times {}^nC_k \begin{bmatrix} n-k \\ c-k \end{bmatrix}\right)$$

*where $\begin{bmatrix} l \\ m \end{bmatrix}$ is the Stirling numbers of the first kind [11] (or the total number of permutations of $l$ symbols, with exactly $m$ cycles, including invariances), and $c$ is the total number of cycles including invariances.*

**Proof.** We have $N = \begin{bmatrix} n \\ c \end{bmatrix}$ permutations of $n$ symbols with exactly $c$ cycles (including invariances). It is known that there are ${}^nC_k$ ways to fix $k$ symbols out of $n$ [11]. Therefore, it is required to subtract all the permutations with $(c-1)$ cycles of length greater than two and one invariance, and $(c-2)$ cycles of length and two invariants etc., from $N$. The principle of inclusion and exclusion can be used to count all the permutations with $c$ cycles of length greater than one. First we subtract the number of permutations ${}^nC_1 \times \begin{bmatrix} (n-1) \\ (c-1) \end{bmatrix}$ from $N$. This underestimates the total number of permutations, since not all of these permutations have invariants. Therefore, we add back ${}^nC_2 \times \begin{bmatrix} (n-2) \\ (c-2) \end{bmatrix}$, etc. Using the principle of inclusion and exclusion [11] the number of permutations can be calculated as $\begin{bmatrix} n \\ c \end{bmatrix} - {}^nC_1 \begin{bmatrix} n-1 \\ c-1 \end{bmatrix} + {}^nC_2 \begin{bmatrix} n-2 \\ c-2 \end{bmatrix} - \ldots + (-1)^n \times {}^nC_c \begin{bmatrix} n-c \\ 0 \end{bmatrix}$. This excludes all the permutations with invariances, resulting in the total number of permutations with exactly $c$ cycles of length at least two. $\square$

**Theorem 1** *The number of nodes $N_d$ at $d$ hops away from the identity permutation in any star graph of size $n!$ is given by*

$$N_d = \sum_{i=0}^{m_1} \left\{ \frac{(n-(q_1+i))}{n} \ {}^{n}C_{q_1+i} \ \boldsymbol{E}^{(p_1-q_1+i)}(n-q_1-i) \right\} + \sum_{i=0}^{m_2} \left\{ {}^{n-1}C_{q_2+i-1} \ \boldsymbol{E}^{(p_2-q_2+i)}(n-q_2-i) \right\}$$

*where*

$$(p_1, q_1) = \begin{cases} (n-d, n-d-1) & \text{when } 1 \le d \le (n-2) \\ (d-n+2, 0) & \text{when } (n-2) < d \le \lfloor \frac{3(n-1)}{2} \rfloor \end{cases}$$

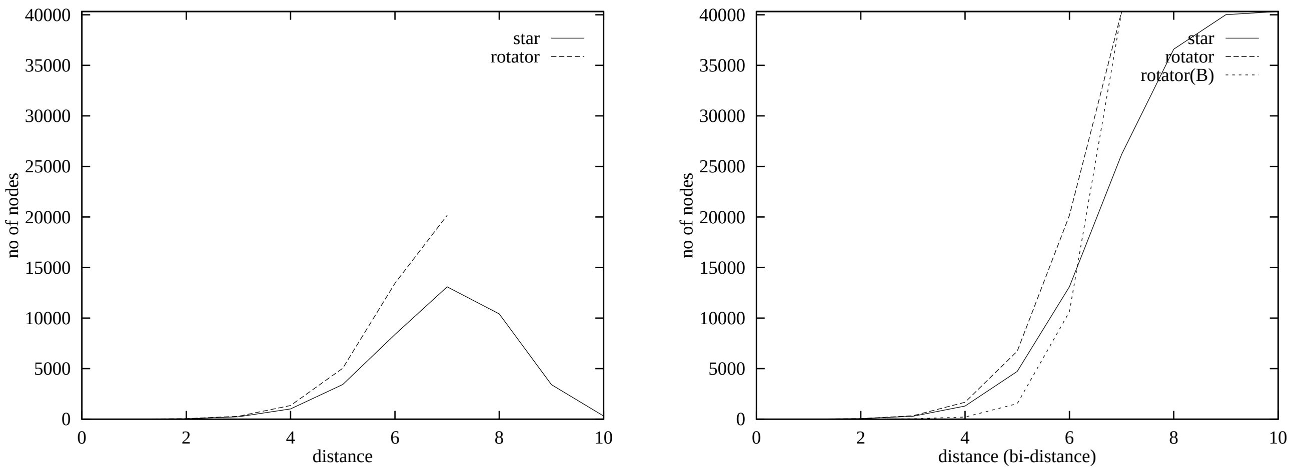
$$(p_2, q_2) = \begin{cases} (n-d+2, n-d+1) & \text{when } 3 \le d \le n \\ (d-n, 0) & \text{when } n < d \le \lfloor \frac{3(n-1)}{2} \rfloor \end{cases}$$

$m_1 = \lfloor \frac{n+q_1-2p_1}{3} \rfloor$, $m_2 = \lfloor \frac{n+q_2-2p_2}{3} \rfloor$, *and the second summation is zero for $d = 1$ and 2.*

**Proof:** Given $n$ and $d$ we calculate the lower bounds for the number of cycles (including invariances) and the number of invariances $(p, q)$ in Appendix A. The calculation of the upper bound $m$ is also given in Appendix A. The upper and lower bounds are calculated from the formula given by Akers and Krishnamurthy [2] for the distance of a permutation from the identity permutation. Lemma 1 gives the number of permutations with exactly $c$ cycles of length greater than one. The permutations of an $\mathcal{S}_n$ are generally classified as two sets one with the first symbol $\pi[1] = 1$ and another with $\pi[1] \neq 1$. Let us consider a pair $(p, q)$, where $p$ is the number of cycles, including invariances and $q$ is the number of invariances. There are ${}^{n}C_q$ ways to fix the $q$ invariances [11], but ${}^{n-1}C_{q-1}$ of them will have $\pi[1] = 1$. Therefore, the number of ways to fix $q$ symbols reduces to ${}^{n}C_q - {}^{n-1}C_{q-1}$, where $\pi[1] \neq 1$. The rest $n-q$ symbols form $\boldsymbol{E}^{(p-q)}(n-q)$ permutations with exactly $p-q$ cycles of length greater than two. Therefore, the total number of permutations of $n$ symbols with $q$ invariances and $p-q$ cycles of length greater than two is $[{}^{n}C_q - {}^{n-1}C_{q-1}] \times \boldsymbol{E}^{(p-q)}(n-q)$. Similarly the number of permutations with $\pi[1] = 1$, $q$ invariances, and $p-q$ cycles of length greater than two is ${}^{n-1}C_{q-1} \times \boldsymbol{E}^{(p-q)}(n-q)$. For every pair of solutions $(p, q)$, the number of permutations are calculated and added to get the total number of permutations. It can be observed from the formula for the distance [2] that, if there are two pairs of solutions $(p_1, q_1)$ and $(p_2, q_2)$, and $q_2 = q_1 + 1$, then $p_2 = p_1 + 2$. □

Recently, Qiu, Fragopoulou, and Akl [12] have proposed another recursive formula for calculating the number of nodes at any distance in $\mathcal{S}_n$. Their formula is entirely different and the proof is much lengthier than our proof. Moreover, we have obtained the formula using the cyclic structure of the permutations. The number of nodes at any distance $d$ from the identity node in $\mathcal{D}_n$ is given by [3, 4], $N_d = \frac{n! \times (n-d)}{(n-d+1)!}$ for $1 \le d \le (n-1)$. It should be noted that $N_d$ in $\mathcal{D}_n$ indicates only the number of nodes $d$ hops away from the identity node. Since $\mathcal{D}_n$ is symmetric there are equal number of nodes from which the identity node is $d$ hops away. However, the number of nodes to which the identity node can communicate in both the directions within $d$ hops is not the same as $N_d$. In [13], the authors present a formula for calculating the number of nodes at *bi-distance* $d$

from any node in $\mathcal{D}_n$. Two nodes $x$ and $y$ in $\mathcal{D}_n$ are at bi-distance $d_b$ when $\max\{\vec{E}_{(x,y)}, \vec{E}_{(y,x)}\} = d_b$. The number of nodes at any distance (and bi-distance) less than the diameter of $\mathcal{S}_n$ and $\mathcal{R}_n$ for




(a) Star vs Rotator: Nodes at distances

(b) Star vs Rotator: Nodes within distances

Figure 3: Comparison of number of nodes at and within various distances in star and rotator graphs

$n = 8$ are plotted in Fig. 3 (a). The number of nodes at various bi-distances from the identity vertex are given in the Fig. 3 as rotator(B), for comparison purposes. While Fig. 3 (a) illustrates the number of nodes exactly at distance $d$, Fig. 3 (b) gives the number of nodes that can be reached within distance $d$ from the identity vertex.

In $\mathcal{D}_n$, the number of nodes increases as the distance increases. However, in $\mathcal{S}_n$, the number of nodes increases from $d = 0$ to $d = (n-1)$ and then decreases from $d = n$ to $d = \lfloor \frac{3(n-1)}{2} \rfloor$ for all values of $n$. This property plays a major role in the average distances of $\mathcal{S}_n$. Though the average distance of $\mathcal{S}_n$ is higher than that of $\mathcal{D}_n$, it is always close to $(n-1)$. Another observation is that $N_d = (n-d) \times \sum_{k=0}^{d-1} N_k$, for $\mathcal{D}_n$. It is known that $N_d = (n-d) \times \frac{n!}{(n-d+1)!}$ [3]. It can be easily shown that $\frac{n!}{(n-d+1)!} = \sum_{k=0}^{d-1} N_k$. This explains the increase in number of nodes as the distance increases. When $d = (n-1)$, $N_{(n-1)} = \frac{n!}{2}$, i.e., one half of the nodes in $\mathcal{D}_n$ are at a distance equal to the diameter of $\mathcal{D}_n$. Since most of the nodes are concentrated at higher distances from the origin, the average diameter of $\mathcal{D}_n$ is close to its diameter and has a lower bound of $(n - 7/4)$ [4].

B. *Minimum Bisection Width*

Minimum bisection width is one of the important parameters in measuring the area complexity of VLSI layouts of multiprocessor interconnection networks [14, 15]. It is known that the binary $n$-cube network has a minimum bisection width of $N$, where $N = 2^n$ is the total number of nodes in the network [14]. Since the layout area of a graph with minimum bisection width $B$ is at least

$\Omega(B^2)$, $k$-ary $n$-cube interconnection networks offer a better bound on the area than the binary $n$-cube [14]. In this subsection we compare the minimum bisection widths of $\mathcal{S}_n$ and $\mathcal{D}_n$.

**Theorem 2** *The minimum bisection width of $\mathcal{S}_n$ with $N = n!$ nodes, where $n \geq 4$ is*

$$B = \begin{cases} N \times \left[\frac{n}{2(n-1)}\right] & \text{if } n \text{ is even} \\ N \times \left[\frac{(n^2-3)}{2n(n-2)}\right] & \text{if } n \text{ is odd} \end{cases}$$

**Proof.** For even values of $n$, $\mathcal{S}_n$ can be viewed as two sets of $\mathcal{S}_{(n-1)}$, each containing $n/2$ $\mathcal{S}_{(n-1)}$ embedded on a two dimensional plane. These two sets can be embedded one on each side of the midpoint. Therefore, only the higher dimensional links connecting $(n-1)$-stars will be cut. For simplicity, we consider the $(n-1)$-substars with the last symbol fixed. The $(n-1)$-substars with the last symbol $1, 2 \ldots n/2$ fixed are placed above the bisection and the substars with the last symbols $n/2+1, n/2+2, \ldots n$ fixed at the last position are placed below the bisection. It can be observed from $\mathcal{S}_n$ that the $(n-1)!$ higher dimensional (bidirectional) links going out of one $\mathcal{S}_{(n-1)}$ are evenly distributed among all other substars. In other words $(n-2)!$ links out of $(n-1)!$ links from an $\mathcal{S}_{(n-1)}$ are connected to each of the other $(n-1)$ $\mathcal{S}_{(n-1)}$. Therefore, bisection of an $\mathcal{S}_n$ for even $n$, will cut $n/2 \times n/2 \times (n-2)!$ bidirectional links. Considering, one bit wide channels in each direction, the bisection width becomes $\frac{n \times n!}{2(n-1)}$. For odd values of $n$, in addition to the links in the $n^{th}$ dimension links in the $(n-1)^{th}$ dimension of one $\mathcal{S}_{(n-1)}$ will also be cut. The arrangement of the substars for odd $n$ can be done as follows; $\frac{(n-1)}{2}$ $\mathcal{S}_{(n-1)}$ with any of the $\frac{(n-1)}{2}$ symbols out of the total $n$ symbols fixed at the last position are placed above and below the midpoint. One $\mathcal{S}_{(n-1)}$ left is placed in between the two sets of $\frac{(n-1)}{2}$ $\mathcal{S}_{(n-1)}$. All the links in the $(n-1)^{th}$ dimension which are cut during the bisection are from this $\mathcal{S}_{(n-1)}$. Also, exactly half of the higher dimensional links from each set of $\frac{(n-1)}{2}$ $\mathcal{S}_{(n-1)}$ will be cut, making the total number of bidirectional links cut

$$\left[\frac{(n-1)}{2} \times \frac{(n-1)}{2}(n-2)!\right] + \left[\frac{(n-1)}{2} \times \frac{(n-1)}{2}(n-3)!\right] + \left[\frac{(n-1)}{2} \times (n-2)!\right]$$

Therefore, the bisection width for odd $n$ is $\frac{n!(n^2-3)}{2n(n-2)}$. $\square$

The minimum bisection of $\mathcal{S}_n$ is shown in Fig 4. It can be observed from Fig. 1 that $B$ is two and four for $\mathcal{S}_2$ and $\mathcal{S}_3$ . Similar arguments in theorem 2, apply to $\mathcal{D}_n$. Only the links in the $n^{th}$ dimension will be cut, when bisecting $\mathcal{D}_n$, for even values of $n$. Similarly, for odd values of $n$, links in the $(n-1)^{th}$ dimension of one $\mathcal{D}_{(n-1)}$ will be cut in addition to the links in the $n^{th}$ dimension connecting other $\mathcal{D}_{(n-1)}$. Since each bidirectional link is considered to have two channels in opposite directions [14], the bisection width of $\mathcal{D}_n$ is the same as that of $\mathcal{S}_n$ for $n \geq 4$. The bisection of $\mathcal{R}_3$ is illustrated in Fig. 5. The bisection width of $\mathcal{R}_3$ is six, since links in all dimensions are cut for $n \leq 3$. Therefore, both the sets of graphs have the same bisection width for $n \geq 4$.

<u>Optimality of the bisection</u> Finding the exact bisection width of a graph with given degree and diameter is a NP-Complete problem.

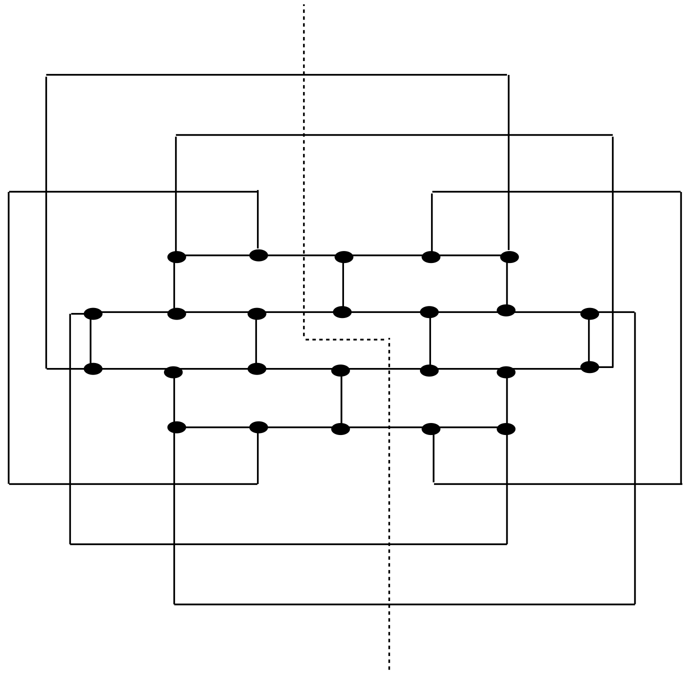


Figure 4: *Minimum bisection of* $\mathcal{S}_4$

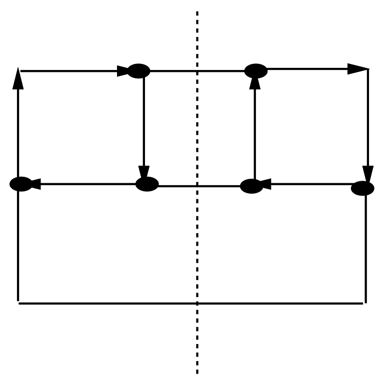


Figure 5: *Minimum bisection of* $\mathcal{R}_3$

## IV. Partitioning and Fault Tolerance

Numerous fault tolerant metrics have been defined in the literature for comparing the fault tolerant properties of interconnection topologies. It is known that $\mathcal{S}_n$ in strongly hierarchical, and it can be partitioned into $n$ copies of disjoint $\mathcal{S}_{(n-1)}$, in $(n-1)$ different ways [2, 7]. The $n$ symbols can be fixed at any of the $(n-1)$ positions from 2 to $n$. This property of star graph is used in many of the applications like broadcasting [16, 17]. Therefore, the number of ways $\mathcal{S}_n$ can be partitioned into $\mathcal{S}_{(n-p)}$ [1] is ${}^{n-1}C_p \frac{n!}{(n-p)!}$. It is known from [1, 2] that $\mathcal{S}_n$ is maximally fault tolerant. When there is a faulty vertex in $\mathcal{S}_n$, it makes ${}^{(n-1)}C_p$ out of ${}^{(n-1)}C_p \frac{n!}{(n-p)!}$, $\mathcal{S}_{(n-p)}$ faulty [1]. Therefore, the minimum number of faults $f(n,p)$, necessary to make every $\mathcal{S}_{(n-p)}$ faulty is $\frac{n!}{(n-p)!}$.

**Theorem 3** *The number of faults $f(n,p)$ necessary to make every $\mathcal{D}_{n-p}$ of $\mathcal{D}_n$ faulty is greater than or equal to $\frac{n!}{(n-p)!}$.*

**Proof.** Since $\mathcal{D}_n$ is hierarchical it be partitioned into $n$ copies of $\mathcal{D}_{(n-1)}$, by fixing any of the $n$ symbols at the last position. Therefore, the number of ways $\mathcal{D}_n$ can be partitioned into $\mathcal{D}_{(n-p)}$ is $\frac{n!}{(n-p)!}$. In $\mathcal{D}_n$, one faulty node makes only one $\mathcal{D}_{(n-p)}$ faulty i.e., any node of $\mathcal{D}_n$ is in only one of the $\mathcal{D}_{(n-p)}$. Therefore, the minimum number of faults necessary to make every $\mathcal{D}_{n-p}$ faulty is $\frac{n!}{(n-p)!}$. □

Therefore, the lower bound on $f(n,p)$ for $\mathcal{D}_n$ is same as that of $\mathcal{S}_n$. For example, taking $n=4$, $f(4,1)$ should be greater than or equal to four to make every $\mathcal{S}_3$ or $\mathcal{R}_3$ faulty.

**Theorem 4** *The minimum number of fault free $\mathcal{S}_{n-p}$ and $\mathcal{D}_{n-p}$ available in the presence of $F$ faults is $\mathcal{S}_{n-p}\{F\} = {}^{n-1}C_p \times \left(\frac{n!}{(n-p)!} - F\right)$ and $\mathcal{D}_{n-p}\{F\} = \frac{n!}{(n-p)!} - F$*

**Proof.** The proof of this theorem is directly from the lower bound of $f(n,p)$ and the number of $\mathcal{S}_{n-p}$ (and $\mathcal{D}_{n-p}$) available. □

Another important measure is the fault diameter $D_f(\mathcal{G}_n)$ of a network. The container length of $\mathcal{S}_n$ is known to be $\lfloor\frac{3(n-1)}{2}\rfloor + 2$ [7]. The exact value of fault diameter of $\mathcal{S}_n$ has been calculated recently by Latifi [10] as $(\lfloor\frac{3(n-1)}{2}\rfloor + 2)$ for $n = 4, 6$ and $(\lfloor\frac{3(n-1)}{2}\rfloor + 1)$ for all other values of $n$. In [4], it is stated that the maximum length of the node disjoint paths between any two vertices in $\mathcal{C}_n$ is $n+1$ for $n \geq 5$. Here we prove that the container length of $\mathcal{R}_n$ is $(n+1)$ for all values of $n > 2$. Before we discuss the container length of $\mathcal{R}_n$ we define $(n-1)$ subtrees of the routing

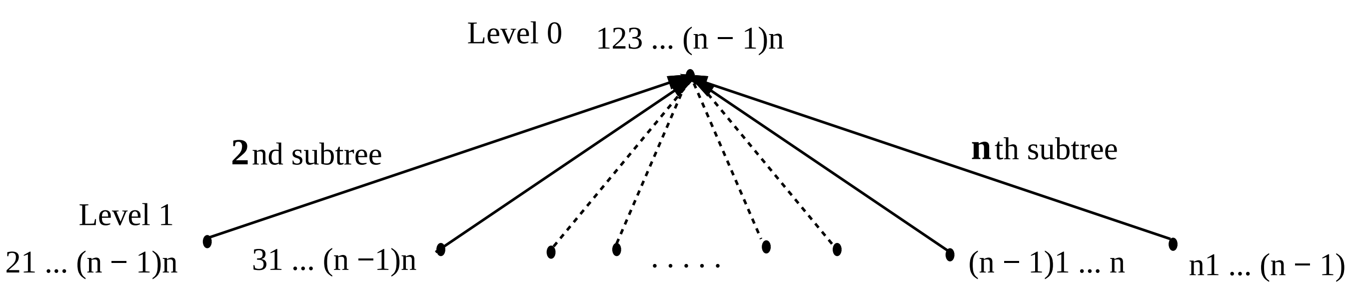



Figure 6: *Routing tree structure of $\mathcal{R}_n$*

tree of $\mathcal{R}_n$. Level zero and level one of the routing tree structure of $\mathcal{R}_n$ is shown in Fig. 6. The $(n-1)$ subtrees of the routing tree are identified as $2^{nd}3^{rd}\dots(n-1)^{th}n^{th}$ subtree. There are $n$ levels in the routing tree and the node at level zero is the identity node. Nodes at $i^{th}$ level $j^{th}$ subtree are of the form $\pi^i_j = x_1x_2\dots x_{i-1}jx_{i+1}\dots x_n$, where $1 \leq i \leq (n-1)$($i$ is the length of the sorted sequence), $2 \leq j \leq n$, i.e., $\pi[i] = j$ for all the permutations. Any message originating from or passing through the permutation $\pi^i_j$, in minimal path routing will always reach the identity permutation through the $j^{th}$ dimension. We use the term "optimal rotation" to denote a rotation of length $r$ applied to a permutation $\pi$, which increases the length of the sorted sequence in $\pi$

by one, where $2 \le r \le n$. Similarly we use the term "non optimal rotation" to denote all other rotations applied to $\pi$.

**Theorem 5** *The container length of $\mathcal{R}_n$ is $(n+1)$.*

**Proof.** In order to have $(n-1)$ node disjoint paths of length at most $(n-1)+g$ from a node $\pi_j^i$ to the identity node in $\mathcal{R}_n$, the permutation $\pi_j^i$ should be sorted into $(n-2)$ distinct permutations of the form $\pi_b^a$, where the distance between $\pi_j^i$ and $\pi_b^a$ is $\le n-a-1+g$, $g > 0$, and $b \ne j$. The routing from $\pi_j^i$ to $\pi_b^a$ should follow disjoint paths and the shortest paths from all $\pi_b^a$ to the identity permutation are guaranteed to be disjoint, since the $(n-2)$ distinct permutations $(\pi_b^a)$ belong to $(n-2)$ different subtrees of the routing tree. Let $r$ be the length of the rotation applied to $\pi_j^i$ in minimal path routing (optimal rotation). This rotation of length $r$ increases the length of the sorted sequence from $(n-i)$ to $(n-i+1)$, where $i \le r \le n$. Rotations of length $l \ge i$ where $l \ne r$, would either result in a permutation with a sorted sequence of length $(n-l+1)$ or $(n-l)$. Similarly, rotations of length $l < i$, and $l \ne r$, would result in a permutation with sorted sequence of length $(n-i)$. If the rotation of length $l > i$, where $l \ne r$, applied to a permutation $\pi_j^i$ results in a permutation with the sorted sequence of length $(n-l+1)$, then this resultant permutation will be in any of the subtrees other than the subtree containing $\pi_j^i$. The distance from these permutations and the identity permutation is at most $(n-1)$. If $l \ge i$ and the length of the sorted sequence is $(n-l)$, then the resultant permutations belong to the $\pi[1]^{th}$ subtree. Similarly, when $l < i$, all the resultant permutations belong to the same subtree as $\pi_j^i$. Now already one generator of non-optimal length has been applied to all these permutations. Thus at most $l-2$ rotations of optimal length can be applied to these permutations without changing the subtree and following node disjoint paths. If any rotation of non-optimal length is applied before the $(l-2)^{th}$ rotation, it brings the first symbol to the position just before the sorted sequence. This would lead to a permutation in another subtree. Now the optimal sorting of this permutation would lead to the identity permutation. The $(n-2)$ subtrees can be selected by rotating any of the $(n-2)$ symbols. Therefore, at most two additional rotations of non-optimal lengths are required to get disjoint paths. This makes the length of the container length of $\mathcal{R}_n$, $(n+1)$. □

The generators used in the node disjoint parallel paths between all the vertices in $\mathcal{R}_4$ and the identity permutation is given in Table I. The exact value of the fault diameter of $\mathcal{D}_n$ is not known. However, from theorem 5 and the diameter of $\mathcal{D}_n$, it can be easily concluded that the fault diameter of $\mathcal{D}_n$ could be either $D_n+1$ or $D_n+2$. Since $D = n-1$ for $\mathcal{D}_n$, the fault diameter of $\mathcal{D}_n$ will always be less than that of $\mathcal{S}_n$. For comparison, let us assume that the fault diameter of $\mathcal{D}_n$ is $(n+2)$, the worst case fault diameter of $\mathcal{D}_n$. A comparison of the fault diameters for $\mathcal{S}_n$ and $\mathcal{D}_n$ is given in Table II. The fault diameter of $\mathcal{D}_n$ is equal to the fault diameter of $\mathcal{S}_n$, only for $n = 3$. For all values of $n > 3$, $D^f(\mathcal{D}_n) < D^f(\mathcal{S}_n)$

TABLE I

NODE DISJOINT PATHS IN $\mathcal{R}_4$

| | |
|---|---|
| $2134 \Rightarrow \{(g_2),(g_3g_2g_3),(g_4g_3g_4g_4)\}$ | $3421 \Rightarrow \{(g_4g_4g_2),(g_2g_3g_4g_3g_4),(g_3g_4g_3g_3)\}$ |
| $3124 \Rightarrow \{(g_3),(g_2g_3g_3g_2),(g_4g_3g_3g_4)\}$ | $4132 \Rightarrow \{(g_4g_2g_3),(g_2g_4g_4g_3g_2),(g_3g_3g_4g_4)\}$ |
| $4123 \Rightarrow \{(g_4),(g_2g_3g_4g_2),(g_3g_4g_4g_4g_3)\}$ | $4231 \Rightarrow \{(g_4g_3g_3),(g_2g_3g_4g_3g_2),(g_3g_4g_4g_4)\}$ |
| $3214 \Rightarrow \{(g_3g_2),(g_2g_3g_3),(g_4g_3g_2g_4)\}$ | $1432 \Rightarrow \{(g_3g_4g_3),(g_2g_3g_3g_4g_4),(g_4g_4g_3g_2)\}$ |
| $4213 \Rightarrow \{(g_4g_2),(g_2g_3g_4),(g_3g_4g_3g_4g_3)\}$ | $2431 \Rightarrow \{(g_4g_4g_3),(g_2g_3g_4g_4g_4),(g_3g_4g_3g_2)\}$ |
| $1324 \Rightarrow \{(g_2g_3),(g_3g_3g_2),(g_4g_4g_3g_4)\}$ | $2143 \Rightarrow \{(g_3g_2g_4),(g_2g_3g_2g_4g_2),(g_4g_3g_4g_3)\}$ |
| $2314 \Rightarrow \{(g_3g_3),(g_2g_3g_2),(g_4g_4g_2g_4)\}$ | $3142 \Rightarrow \{(g_4g_2g_4),(g_2g_4g_4g_4g_2),(g_3g_3g_4g_3)\}$ |
| $4312 \Rightarrow \{(g_4g_3),(g_2g_4g_4),(g_3g_4g_3g_4g_2)\}$ | $1243 \Rightarrow \{(g_3g_3g_4),(g_2g_3g_3g_4g_2),(g_4g_4g_4g_3)\}$ |
| $1423 \Rightarrow \{(g_2g_4),(g_3g_4g_2),(g_4g_4g_3g_3)\}$ | $3241 \Rightarrow \{(g_4g_3g_4),(g_2g_3g_4g_4g_2),(g_3g_4g_4g_3)\}$ |
| $2413 \Rightarrow \{(g_3g_4),(g_2g_4g_2),(g_4g_4g_2g_3)\}$ | $1342 \Rightarrow \{(g_3g_4g_4),(g_2g_3g_3g_4g_3),(g_4g_4g_4g_2)\}$ |
| $3412 \Rightarrow \{(g_4g_4),(g_2g_4g_3),(g_3g_4g_3g_3g_2)\}$ | $2341 \Rightarrow \{(g_4g_4g_4),(g_2g_3g_4g_4g_3),(g_3g_4g_4g_2)\}$ |
| $4321 \Rightarrow \{(g_4g_3g_2),(g_2g_3g_4g_3g_3),(g_3g_4g_3g_4)\}$ | |

TABLE II

COMPARISON OF FAULT DIAMETERS OF $\mathcal{S}_n$ AND $\mathcal{D}_n$

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_n$ | 4 | 6 | 7 | 9 | 10 | 11 | 13 | 14 | 16 |
| $\mathcal{D}_n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Another interesting problem to investigate is the incomplete or clustered networks [18]. We restrict our analysis to the problem of removing sub-networks of size $(n-1)!$ from $\mathcal{G}_n$. The incomplete network $\mathcal{G}_{n-1}^m$ is defined as the network obtained from $\mathcal{G}_n$ by removing $(n-m)$ of its $\mathcal{G}_{(n-1)}$. The clustered star network [18] and the clustered rotator graphs [13] have been studied in the literature. A variation of this problem for an arbitrary number of nodes in star [19] and rotator [20] have also been studied in the literature. The $\mathcal{R}_3^2$ is shown in Fig. 7. $\mathcal{S}_3^2$ can be obtained similarly

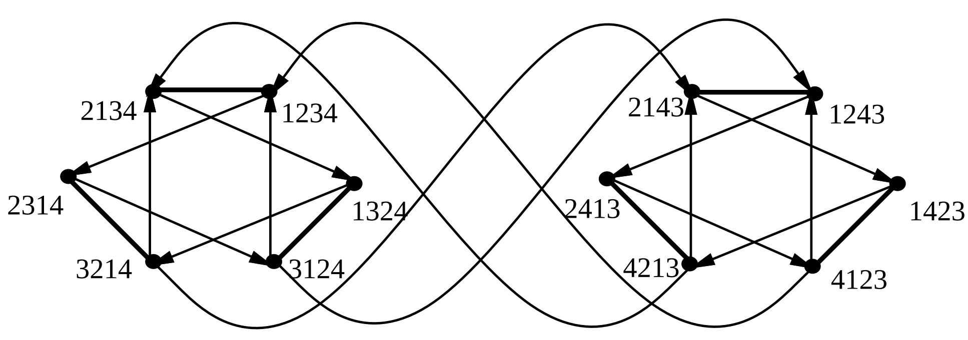



Figure 7: *Incomplete rotator ($\mathcal{R}_3^2$)*

with two links the the fouth dimension connecting the two 3-substars (see Fig. 1). The study of clustered networks can be used in processor allocation and fault tolerant applications. A new class of networks with variable number of nodes with properties similar to the original network can also be obtained from these clustered networks. The clustered rotator graphs have been shown to be Hamiltonian for all values of $m$, where $1 \leq m \leq (n-1)$ [13]. Similar network obtained for the star graph [10] is not Hamiltonian for all values $m$. $\mathcal{S}_n^m$ is shown to be Hamiltonian for $m = 4$ and $m = 3k, k \neq 2$. It can be observed from Fig. 1 and Fig. 7 that $\mathcal{R}_3^2$ is Hamiltonian, whereas $\mathcal{S}_3^2$ is not Hamiltonian. However, the diameter of $\mathcal{S}_{n-1}^m$ is the same as the diameter of $\mathcal{S}_n$ [18] for all values of $n$. Since there is a unique shortest path between any two vertices the diameter of $\mathcal{R}_{n-1}^m$ higher than the diameter of $\mathcal{R}_n$.

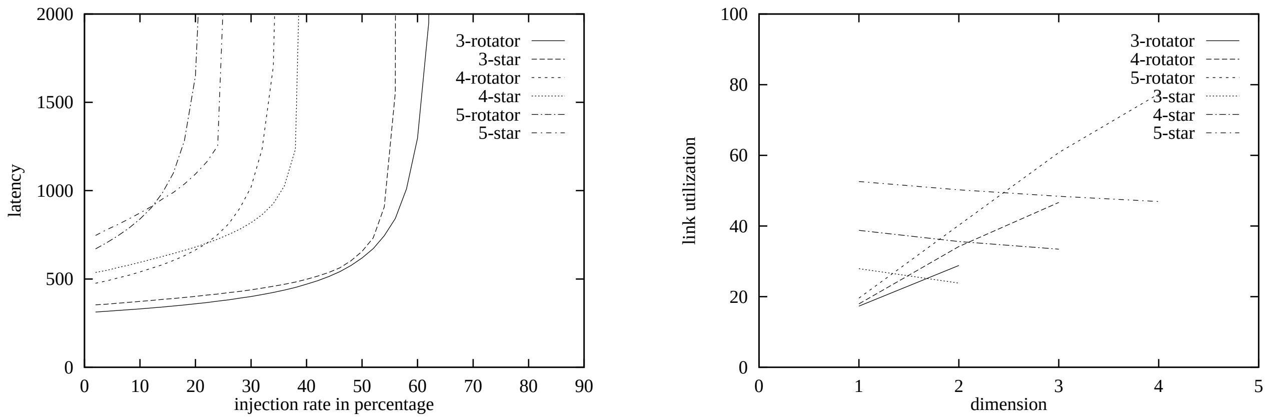
## V. Information Dissemination and Embedding

Routing from one node to another node, one to all broadcasting, and all to all broadcasting are the three important information dissemination problems that often arise in most of the applications involving parallel computations. In this section we analyze the information dissemination problems in $\mathcal{S}_n$ and $\mathcal{D}_n$.

### A. *Routing*

In any Cayley graph, routing from a vertex $x$ to vertex $y$ is simply the sorting of the permutation $xy^{-1}$ to the identity permutation using the generators of the graph [2, 3]. The greedy approach presented in [2] results in optimal path routing in $\mathcal{S}_n$. Similarly the optimal routing algorithm for $\mathcal{D}_n$ is also simple [3, 4]. The disjoint paths between any two vertices of $\mathcal{S}_n$ has been investigated by many authors [21, 7]. The length of the disjoint paths between any two vertices $x$ and $y$ can be either $c + m$ or $c + m + 2$ or $c + m - 2$, depending on the first symbol of the permutation $xy^{-1}$, where $c$ is the number of cycles in $xy^{-1}$ of length at least two and $m$ is the total number of misplaced symbols in $xy^{-1}$. We have presented the node disjoint paths in $\mathcal{D}_n$ in an earlier section. It is known that only one path out of the $(n-1)$ node disjoint paths between any two vertices in $\mathcal{D}_n$ is optimal. In Fig. 8(a) simulation results of comparison of latencies in minimal path routing of $\mathcal{S}_n$ and $\mathcal{R}_n$, for $n = 3, 4$ and 5 are given. It can be observed from the figure that $\mathcal{R}_n$ performs better than $\mathcal{S}_n$ under low traffic, but the performance degrades under heavy traffic. Fig. 8(b) illustrates the percentage of links used in each dimension, for a specific value of traffic intensity. It can be observed from the figure that the higher dimensions in $\mathcal{R}_n$ are used more number of times than the lower dimensions, whereas the message distribution is almost uniform in $\mathcal{S}_n$. The edge symmetry of star graph ensures uniform message distribution under heavy traffic. The results shown for the minimal path routing in $\mathcal{S}_n$ selects the lowest possible dimension if multiple paths are available and the link is free i.e. when more than one optimal path is available from the source or intermediate node to the destination node the dimensions leading to the optimal paths are probed from the lowest dimension.

It should be noted that, the number of nodes in $\mathcal{S}_n$ which have an unique shortest path from the identity permutation $N_u = \sum_{k=2}^{n} \frac{(n-1)!}{(n-k)!}$. There is an unique shortest path between vertex $P = p_1 p_2 \ldots p_n$ and the identity permutation if $P$ contains only one cycle of length at least two and $p_1 \neq 1$. Therefore, this is simply the total number of permutations with one cycle of length




(a) Star vs Rotator: Latency

(b) Star vs Rotator: Link Utilization

Figure 8: Comparison of average latency and link utilization in rotator and star graphs for $n = 8$

$2, 3, \ldots$ and $n$, excluding the permutations with the symbol one at the first place. The number of permutations with one cycle of length $k$ and $(n-k)$ invariances can be calculated from Cauchy's formula [22] as $\frac{n!}{(n-k)!k}$, where $2 \leq k \leq n$. The number of permutations with $p_1 = 1$ and one cycle of length $k$ is $\frac{(n-1)!}{(n-k-1)!k}$, and this can be subtracted from $\frac{n!}{(n-k)!k}$, for every value of $k$, to calculate the total number of nodes have unique shortest path. This leads to a total number of $\sum_{k=2}^{n} \frac{(n-1)!}{(n-k)!}$ nodes. Whereas in $\mathcal{D}_n$, when messages are routed from every node to the identity node the number of times $i^{th}$ dimension is used is the same as the number of nodes at distance $i$ away from the identity node [13]. It is known from section two that number of nodes increases as the distance from the origin increases. This explains the maximum use of the higher dimensional links in minimal path routing.

The spanning tree structure of $\mathcal{S}_n$ is analyzed in [7]. They present a balanced spanning tree and calculate the balance factor to be $BF(Star) = \frac{\sum_{k=1}^{(n-1)} k!}{(n-1)!}$. The value of $BF(Star)$ converges to 1 as $n$ increases. However, the height of the spanning tree structure obtained is $2n-3$, which is not optimal. A balanced communication can be achieved by a careful selection of the routing path. It can be observed from Fig. 8 (a) that the average link utilization for the lower dimensions are slightly higher than that of the higher dimensions. This is due to the order in which the dimensions of multiple optimal paths are probed to find whether they are free. Since the probing is always done from the lower dimension to the higher dimension and the probability of a lower dimension link being free is higher than that of the higher dimensional links, the link utilization is slightly high for the lower dimensions. Almost uniform message distribution can be achieved by randomly selecting the probing order. For $\mathcal{D}_n$, Faber, Moore, and Chen [4] note that a very simple routing scheme called *natural routing* can be used. Since the average distance of $\mathcal{D}_n$ is close to the diameter, the natural routing chooses any path between any two vertices that is of length less than the diameter of $\mathcal{D}_n$. However, in $\mathcal{S}_n$, the average distance is close to $(n-1)$. Therefore, if the diameter is used as a bound, similar to $\mathcal{D}_n$ the average distancer and hence the average latency will increase. Since the node disjoint parallel paths in these graphs are known, and the container length of these graphs is greater than the shortest distance by only a small constant, comparison of non-optimal routing would also be interesting to analyze.

### B. *Broadcasting*

An optimal algorithm for broadcasting in any network with $N$ nodes must take at least $\Omega(\log_2 N)$ steps. Since $\mathcal{G}_n$ has $n!$ vertices, optimal broadcasting in $\mathcal{G}_n$ should take $O(n \log_2 n)$ steps. Mendia and Sarkar [16] proposed an optimal broadcasting algorithm for $\mathcal{S}_n$. They develop an $O(n^2)$ algorithm for broadcasting in $\mathcal{S}_n$ and improve that algorithm to $O(n \log_2 n)$ by improving some intermediate steps. In can be observed from the structure of $\mathcal{D}_n$ that one-to-all broadcasting can be done in $O(n^2)$ trivially. Consider the vertex $12\ldots n$ in $\mathcal{R}_n$ which has the message to be broadcast. In phase I the message is sent to all the other $(n-2)$ nodes which have the symbols $2, 3$, and $(n-2)$ at the first position. This phase takes exactly $(n-1)$ steps i.e.

$$12\ldots n \overset{g_n}{\Rightarrow} 23\ldots n1 \overset{g_n}{\Rightarrow} \ldots\ldots \overset{g_n}{\Rightarrow} (n-1)n\ldots(n-3)(n-2)$$

At the end of first phase, in the second phase one additional step is required to broadcast the message to at least one node in all the $\mathcal{R}_{(n-1)}$ in the network. This process can be repeated recursively and the problem can be reduced to a broadcast in $\mathcal{R}_2$. This scheme requires $n(n+1)/2$ steps which is $O(n^2)$. Optimal algorithm for broadcasting in $\mathcal{D}_n$ is not known. The same improvements made in the $O(n^2)$ algorithm for $\mathcal{S}_n$ cannot be made in the $O(n^2)$ algorithm for $\mathcal{D}_n$. The interconnection structure of $\mathcal{S}_n$ allows to broadcast a message from a node of $\mathcal{S}_n$ to at least one node in every other $\mathcal{S}_{n-1}$ in $\lceil \log_2 n \rceil$ steps. The generators in $\mathcal{D}_n$ only allow a message to broadcast from one node to at least one node in every other $\mathcal{D}_{n-1}$ in $O(n)$ steps. Therefore, an entirely different broadcasting technique is necessary to broadcast optimally in $\mathcal{D}_n$.

### C. *Embedding*

Embedding other interconnection structures in $\mathcal{S}_n$ have been studied by many authors [23, 24, 25]. It has been shown [23, 25] that multidimensional meshes can be embedded in $\mathcal{S}_n$ with dilation three and expansion one. It is also conjectured [23] that a dilation two embeddings of meshes on star exist. Nigam and Krishnamurthy [24] present dilation two (expansion $(2^d+1)!/2^d$), and dilation three (expansion $d!/2^d$) embedding of hypercubes in star graphs. Due to the directed nature of $\mathcal{D}_n$ the optimal dilation of the embedding of an undirected mesh in $\mathcal{D}_n$ is shown to have a lower bound of $\lceil n/2 \rceil$ [13]. Both the sets of graphs $\mathcal{S}_n$ and $\mathcal{D}_n$ have multiple Hamiltonian circuits [24, 23, 13].

However, dilation of the embedding alone cannot be considered as a measure of good embedding. Since the dilation of embedding of mesh in star graph is three, there is a possibility that some nodes in $\mathcal{S}_n$ are used as intermediate nodes for many different pairs of adjacent nodes in the mesh [26]. This leads to congestion as the number of messages routed through this node in mesh increases. Therefore, the communication cost of the embedding is also important in real world applications. Qiu, Meijer, and Akl [26] note that, the node $123\ldots n$ is used by $n-2$ pairs of adjacent nodes in the mesh, using the embedding algorithm in [23]. This results in a communication cost of at least $\Omega(n)$ . They present a $(n-1)$-dilation, $(n-1)$-expansion embedding of meshes in star graphs and show that the communication cost is comparable to that of dilation three, expansion one embedding. Since the diameter of $\mathcal{D}_n$ is $(n-1)$, any arbitrary embedding of meshes on $\mathcal{D}_n$ with dilation $(n-1)$ and expansion one would have low communication cost on the average. However, better embeddings can be found for meshes of restricted dimensions with low average dilation [13].

TABLE III
COMPARISON OF $\mathcal{S}_n$ AND $\mathcal{D}_n$

| *Description* | $\mathcal{S}_n$ | $\mathcal{D}_n$ |
|---|---|---|
| Nodes | n! | n! |
| Diameter | $\lfloor\frac{3(n-1)}{2}\rfloor$ | n - 1 |
| Average Distance | $n+\frac{2}{n}+\sum_{i=1}^{n}\frac{1}{i}-4$ | $n+1-e+\sum_{i=n-1}^{\infty}\frac{1}{i!}$ |
| Links | Undirected | Directed |
| Symmetry | Vertex and Edge Symmetric | Vertex Symmetric |
| Hierarchical | Strongly Hierarchical | Hierarchical |
| Optimal Paths | Multiple | Unique |
| Number of $\mathcal{G}_{(n-p)}$ | ${}^{n-1}C_p\frac{n!}{(n-p)!}$ | $\frac{n!}{(n-p)!}$ |
| Number of $\mathcal{G}_{n-p}\{F\}$ | ${}^{n-1}C_p \times \frac{n!-F\times(n-p)!}{(n-p)!}$ | $\frac{n!-F\times(n-p)!}{(n-p)!}$ |
| Incomplete Network | Not Hamiltonian | Hamiltonian |
| Container Length | $\lfloor\frac{3(n-1)}{2}\rfloor+2$ | $(n+1)$ |
| $D_n^f$ | $\lfloor\frac{3(n-1)}{2}\rfloor+1$ | $\leq(n+1)$ |
| Message Distribution | Almost Uniform | Non-uniform |

## VI. SUMMARY AND CONCLUSION

A summary of the properties in which the two sets of graphs $\mathcal{S}_n$ and $\mathcal{D}_n$ differ is given in Table III. The parameters $f(n,p)$ and minimum bisection width are shown to be the same for both $\mathcal{S}_n$ and $\mathcal{D}_n$. Considering unidirectional physical connections, the number of unidirectional links used is also same for both the sets of graphs for the same value of $n$. Directed interconnection networks have some advantages like ease of construction, simple link level protocol, etc. Therefore, $\mathcal{D}_n$ is an attractive alternative if directed interconnections are preferred. Apart from its low diameter and average distance, the number of nodes at any distance in $\mathcal{D}_n$ is easy to calculate compared to $\mathcal{S}_n$. The distribution of nodes at various distances also differ in $\mathcal{D}_n$ from $\mathcal{S}_n$. The average distance of $\mathcal{D}_n$ is close to its diameter $(n-1)$. Though the average diameter of $\mathcal{S}_n$ is higher than that of $\mathcal{D}_n$, it is also close to $(n-1)$, i.e., compared to its diameter $(\frac{3(n-1)}{2})$, $\mathcal{S}_n$ has a better average dilation than $\mathcal{D}_n$. Recent advances in optical interconnection technology indicates that multiple channels can be realized in a single physical connection. The directed Cayley graphs are suitable for such applications, due to the fact that optical links are directed, and there is no redundant paths of minimal length. Applications with mesh type of computations on both star graphs and directed Cayley graphs involve some additional cost. Indigenous algorithms need to be developed for both the architectures in order to execute mesh and binary $n$-cube type of regular communications. In addition to the diameter and average diameter, the fault diameter of $\mathcal{D}_n$ is lower than that of $\mathcal{S}_n$. This is one of the attractive fault tolerant properties of $\mathcal{D}_n$. Most of the disadvantages of $\mathcal{D}_n$ are due to its lack of edge symmetry. The edge and node symmetry of star graph is one of its major advantage over $\mathcal{D}_n$ as it helps in broadcasting and fault tolerant applications. Non uniform message distribution in $\mathcal{D}_n$ leads to the saturation of communication links under heavy traffic. However, the routing can be further improved by increasing the number of non optimal paths chosen under heavy traffic. It would be interesting to study routing, sorting, and broadcasting in $\mathcal{D}_n$.

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## Appendix A: Star Graph Parameters

The formula for calculating the distance between a vertex $\pi$ and the identity node is given by [2] $d = n+x-2y$, if $\pi[1] = 1$, and $d = n+x-2y-2$ if $\pi[1] \neq 1$, where $x$ is the total number of cycles in $\pi$, including invariances and $y$ is the number of invariances. It is required to find lower and upper bounds for the pair of solutions $(x_1, y_1)$ and $(x_2, y_2)$ of the equations $x_1 - 2y_1 = d-n+2 = C_1$ and $x_2 - 2y_2 = d - n = C_2$. We use $x$ to denote both $x_1$ and $x_2$. Similarly $y$ is used to denote both $y_1$ and $y_2$. The solution of the above two equations are subject to the conditions: *(i)* $0 \leq y < x < n$, *(ii)* $x = y$, only when $x = n$, and *(iii)* $(x - y) \leq \lfloor \frac{(n-y)}{2} \rfloor$. Condition *(i)* states that the number of invariances is always greater than or equal to zero and less than the total number of cycles $x$ for values of $x$ less than $n$. When the number of cycles is equal to the number of symbols, all the cycles are invariances as given by *(ii)*. Since the number of cycles $(x - y)$ of length greater than one, cannot be less than $\lfloor \frac{(n-y)}{2} \rfloor$, condition *(iii)* is also required.

**A: For $x_1^{min}$ and $y_1^{min}$**

We consider two cases for $d \leq (n-2)$, and $d > (n-2)$. When $d \leq (n-2)$, $C_1 \leq 0$, and substituting $|C_1|$ for $y_1$ would give $x_1 = |C_1|$, which is not a valid solution since $y_1$ should be less than $x_1$ for values of $x_1 < n$. Similarly for values of $y_1$ less than $|C_1|$, $x_1$ will be less than $y_1$, which is not valid. Therefore, the next possible value for $y_1 = |C_1|+1$, which makes the value of $x_1 = |C_1|+2$. This makes the lower bound for $(x_1, y_1)$ is $(|C_1| + 2, |C_1| + 1)$. When $d > (n - 2)$, $C_1$ is positive and hence the lower bound for $(x_1, y_1)$ is $(C_1, 0)$.

**B: For $x_2^{min}$ and $y_2^{min}$**

Similarly, solving the equations $x_2, y_2$ and $C_2$, would give the lower bounds $(|C_2| + 2, |C_2| + 1)$, $(C_2, 0)$, for $d \leq n$, and $d > n$ respectively. However, for $d = 1$, $x_2 = |C_2| + 2 > n$, which is not a valid solution. And, when $d = 2$, $x_2 = |C_2| + 2 = n$, and $y_2 = (n-1)$, which violates the condition that $y = n$, when $x = n$. Therefore, the value of $d$ ranges from 3 to $n$, as there is no valid solutions for $d = 1$ and 2.

The upper bounds for $x$ and $y$ can be calculated from condition *(iii)* and the lower bounds $x^{min}$ and $y^{min}$. It can be observed from the equations that, all $(x, y)$ of the form $(x^{min} + 2m, y^{min} + m)$, where $i \geq 0$ are valid solutions. Substituting these for *(iii)* gives $2(x^{min} + 2m) \leq \lfloor n + (y^{min} + m) \rfloor$ $\Rightarrow m \leq \lfloor \frac{n + y^{min} - 2x^{min}}{3} \rfloor$, $x^{max} = 2m$, and $y^{max} = m$.