# Predicate Logic

CSE 191, Class Note 02:
Predicate Logic
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# From Proposition to Predicate

$3 + 2 = 5$ is a proposition. But is $X + 2 = 5$ a proposition?

- Because it has a variable $X$ in it, we cannot say it is T or F.
- So, it is not a proposition. It is called a predicate.

**Definition**

A predicate is a function. It takes some variable(s) as arguments; it returns either True or False (but not both) for each combination of the argument values.

- In contrast, a proposition is not a function. It does not have any variable as argument. It is either True or False (but not both).
- The variables are always associated with a universe (or domain) of discourse, which tells us what combinations of the argument values are allowed.

## Predicate vs. Proposition

- Suppose $P(x)$ is a predicate, where the universe of discourse for $x$ is $\{1, 2, 3\}$. Then $P(x)$ is not a proposition, but $P(1)$ is a proposition.
- In general, a predicate is not a proposition. But when you assign values to all its argument variables, you get a proposition.

**Example:**

$P(x, y)$ : "$x + 2 = y$" is a predicate.

- It has two variables $x$ and $y$;
- Universe of Discourse: $x$ is in $\{1, 2, 3\}$; $y$ is in $\{4, 5, 6\}$.

- $P(1, 4)$ : $1 + 2 = 4$ is a proposition (it is F);
- $P(2, 4)$ : $2 + 2 = 4$ is a proposition (it is T);
- $P(2, 3)$: meaningless (in this example), because 3 is not in the specified universe of discourse for $y$.



## Example

**Example**

Suppose $Q(x, y)$ : "$x + y > 4$", where the universe of discourse is all integer pairs.

Which of the following are predicates? Which are propositions?

- $Q(1, 2)$
- $Q(1000, 2)$
- $Q(x, 2)$
- $Q(1000, y)$
- $Q(x, y)$

## Universal quantification

Universal quantification is another way of converting a predicate into a proposition.

### Definition

Suppose $P(x)$ is a predicate on some universe of discourse.

- The universal quantification of $P(x)$ is the proposition: "$P(x)$ is true for all $x$ in the universe of discourse".
- We write $\forall x P(x)$, and say "for all $x$, $P(x)$".
- $\forall x P(x)$ is TRUE if $P(x)$ is true for every single $x$.
- $\forall x P(x)$ is FALSE if there is an $x$ for which $P(x)$ is false.



## Universal quantification example

### Example:

$P(x)$ : "$x + 2 = 5$, universe of discourse: $\{1, 2, 3\}$.

$\forall\ x\ P(x)$ means: "for all $x$ in $\{1, 2, 3\}$, $x + 2 = 5$".
In other words, it means: "1+2=5, 2+2=5, and 3+2=5", which is false.
So it is a proposition.

### More example:

$A(x)$ : "$x = 1$" $B(x)$ : "$x > 2$" $C(x)$ : "$x < 2$"
Universe of discourse is $\{1, 2, 3\}$.

True or False?
a) $\forall\ x\ (C(x) \to A(x))$
b) $\forall\ x\ (C(x) \lor B(x))$

## Existential quantification

Existential quantification is yet another way of converting a predicate into a proposition.

### Definition

Suppose $P(x)$ is a predicate on some universe of discourse.

- The existential quantification of $P(x)$ is the proposition: "$P(x)$ is true for some $x$ in the universe of discourse."
- We write $\exists\, x\; P(x)$, and say "there exists $x,\; P(x)$".
- $\exists\, x\; P(x)$ is TRUE if $P(x)$ is true for any $x$.
- $\exists\, x\; P(x)$ is FALSE if for every $x$, $P(x)$ is false.



## Existential quantification example

### Example:

$P(x)$ : "$x + 2 = 5$", universe of discourse: $\{1, 2, 3\}$.

$\exists\, x\; P(x)$ means: "for some $x$ in $\{1, 2, 3\}$, $x + 2 = 5$".
In other words, it means: "$1 + 2 = 5$, or $2 + 2 = 5$, or $3 + 2 = 5$" which is true. So it is a proposition.

### More examples on existential quantifier:

$A(x)$ : "$x = 1$", $B(x)$ : "$x > 5$", $C(x)$ : "$x < 5$".
Universe of discourse is $\{1, 2, 3\}$.

True or False?
a) $\exists\, x\; (C(x) \to A(x))$
b) $\exists\, x\; B(x)$

## More examples

### Example:

$A(x)$: "$x$ lives in Amherst."
$B(x)$: "$x$ is a CSE 191 student."
$C(x)$: "$x$ has a good GPA."
$D(x)$: "$x$ majors in computer science."
Universe of discourse: all UB students.

- All CSE 191 students have good GPA: $\forall\, x\ (B(x) \rightarrow C(x))$
- No CSE 191 student lives in Amherst: $\neg\, \exists\, x\ (B(x) \wedge A(x))$
- CSE 191 students who do not live in Amherst major in computer science: $\forall x\ (B(x) \wedge \neg\, A(x) \rightarrow D(x))$



## Quantifier negation

### Example:

Consider the following two propositions:

- Not every UB student majors in computer science: $\neg\, \forall x\ D(x)$.
- There is UB student who does not major in computer science: $\exists\, x\ \neg\, D(x)$.

Do these two statements have the same meaning? YES.

### Example:

Consider the following two propositions:

- There is no UB student living in Amherst: $\neg\, \exists\, x\ A(x)$.
- Every UB student lives in a town other than Amherst: $\forall\, x\ \neg\, A(x)$.

Do these two statements have the same meaning? YES.

## Quantifier negation

### Quantifier negation: In general we have:

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

Where $\equiv$ means logical equivalence as we have seen.

### Rule: to negate a quantifier:

- move the negation to the inside;
- and switch $\exists$ to $\forall$, $\forall$ to $\exists$.



## Nested Quantifier

- Sometimes we have more complicated logical formulas that require nested quantifiers.
- "Nested" means one quantifier's scope contains another quantifier.

- $\forall x \forall y P(x, y) = T$ if $P(x, y) = T$ for all $x$ and $y$.
- $\forall x \exists y P(x, y) = T$ if for any $x$ there exists $y$ such that $P(x, y) = T$.
- $\exists x \forall y P(x, y) = T$ if there exists $x$ such that $P(x, y) = T$ for all $y$.
- $\exists x \exists y P(x, y) = T$ if these exists $x$ and there exists $y$ such that $P(x, y) = T$.

## Examples for translating logic formulas to sentences

### Example:

Suppose $P(x, y)$ means: "$x$ (the first variable in $P(*.*)$ loves $y$ (the second variable in $(P(*, *)$"), where the domain of $x$ is the students in this class and the domain of $y$ is the courses offered by UB CSE.

Note: The variable is defined by its position in $P(*, *)$. We might use different names for them. So when we write $P(y, w)$, we are saying $y$ loves $w$.

- $P(Alice, CSE191)$: Alice loves CSE191.
- $\exists\, y\; P(Alice, y)$: Alice loves a UB CSE course.
- $\exists\, x\; (P(x, CSE191) \wedge P(x, CSE250))$:
  A student in this class loves both CSE 191 and CSE 250.
- $\exists\, x\, \exists\, y\, \forall\, z\; ((x \neq y) \wedge (P(x, z) \rightarrow P(y, z)))$:
  There are two different students $x$ and $y$ in this class such that if $x$ loves a UB CSE course, then $y$ loves it as well.



## Example for translating sentences to logic formulas

Now we consider translations in the other direction.

### Example:

Suppose $P(x, y)$ means: "$x$ loves $y$", where the domain of $x$ is the students in this class and the domain of $y$ is the courses offered by UB CSE.

- Every UB CSE course is loved by some student in this class:
  $\forall\, y\, \exists\, x\; P(x, y)$.
- No student in this class loves both CSE 191 and CSE 250:
  $\neg\, \exists\, x\; (P(x, CSE191) \wedge P(x, CSE250))$.

# Nested quantifier: examples

## Example:

$A(x)$: "$x$ lives in Amherst."
$C(x, y)$: "$x$ and $y$ are friends."
$D(x)$: "$x$ majors in computer science."
Universe of discourse: all UB students.

There is a computer science major who has a friend living in Amherst:
$\exists\, x\ (D(x) \land \exists\, y\ (A(y) \land C(x, y)))$

## Example:

$A(x)$: "$x$ lives in Amherst."
$B(x)$: "$x$ is a CSE 191 student."
$C(x, y)$: "$x$ and $y$ are friends."
$D(x)$: "$x$ majors in computer science."
Universe of discourse: all UB students.

Either there is a computer science major who has a friend living in Amherst, or all CSE 191 students major in computer science:
$\exists\, x\ (D(x) \land \exists\, y\ (A(y) \land C(x, y))) \lor \forall\, x\ (B(x) \to D(x))$



# Compare examples

## Example:

$A(x)$: "$x$ lives in Amherst."
$B(x)$: "$x$ is a CSE 191 student."
$C(x, y)$: "$x$ and $y$ are friends."
$D(x)$: "$x$ majors in computer science."
Universe of discourse: all UB students.

Consider the following two statements:

- $\forall\, x\ \exists y(C(x, y) \land B(y))$:
  All UB students have friends taking CSE191.
- $\exists\, y\ \forall\, x\ (C(x, y) \land B(y))$:
  There is a UB student who is the friend of all UB students and takes CSE 191.

Do they mean the same thing? NO.

## Order Matters

Difference in orders of quantifiers may lead to difference in truth values.

### Example:

Let the universe of discourse be pairs of real numbers.

- $\forall\, x \,\exists\, y\ (y > x) \equiv$ True
  But
- $\exists\, y \,\forall\, x\ (y > x) \equiv$ False

### In general:

- $\forall x \forall y P(x,y) = T$ iff $\forall y \forall x P(x,y) = T$.
- $\exists x \exists y P(x,y) = T$ iff $\exists y \exists x P(x,y) = T$.
- If $\exists x \forall y P(x,y) = T$, then $\forall y \exists x P(x,y) = T$.
- If $\forall y \exists x P(x,y) = T$, $\exists x \forall y P(x,y)$ might be F.



## Compare examples

### Example:

$A(x)$: "$x$ lives in Amherst."
$B(x)$: "$x$ is a CSE 191 student."
$C(x,y)$: "$x$ and $y$ are friends."
$D(x)$: "$x$ majors in computer science."
Universe of discourse: all UB students.

- $\exists\, x\ D(x) \wedge\ \exists\, x\, \exists\, y\ (A(y) \wedge C(x,y))$:
  There is a computer science major and there is a UB student who has a friend living in Amherst.
  Compared with:
- $\exists\, x\ (D(x) \wedge \exists\, y\ (A(y) \wedge C(x,y)))$:
  There is a computer science major who has a friend living in Amherst.

The meaning of the two statements are different. This is because the scopes of the variable $\exists\, x$ in the two formula are different.

## Scope matters

In general, difference in scopes of quantifiers can lead to difference in truth values.

Example:

- $\forall x\ (P(x) \vee \neg P(x)) \equiv$ True
  But the truth value of
- $\forall x\ P(x) \vee \neg P(x)$
  depends on the truth value of free variable $x$. (The second $x$ is not bounded by any quantifier.) So this formula is not even a proposition!



## Nested quantifier: more examples

Example:

$A(x)$: "$x$ lives in Amherst."
$B(x)$: "$x$ is a CSE 191 student."
$C(x)$: "$x$ has a good GPA."
$D(x)$: "$x$ majors in computer science."
Universe of discourse: all UB students.

If all computer science majors have friends taking CSE191, then there are a pair of friends both taking CSE191.
$\forall x\ (D(x) \rightarrow \exists y\ (B(y) \wedge C(x,y))) \rightarrow \exists x\ \exists y\ (B(x) \wedge B(y) \wedge C(x,y))$

# Rules for quantifier scope

**We have the following important rules:**

$$
\begin{aligned}
\forall x\ (P(x) \wedge Q(y)) &\equiv \forall x\ P(x) \wedge Q(y) \\
\forall x\ (P(x) \vee Q(y)) &\equiv \forall x\ P(x) \vee Q(y) \\
\exists x\ (P(x) \wedge Q(y)) &\equiv \exists x\ P(x) \wedge Q(y) \\
\exists x\ (P(x) \vee Q(y)) &\equiv \exists x\ P(x) \vee Q(y)
\end{aligned}
$$

In general, as long as a subformula does not contain a quantifier's bound variable, you can arbitrarily put it into the scope of that quantifier, or take it out of the scope of that quantifier.