CSE 191, Class Note 02:
Predicate Logic
Computer Sci & Eng Dept
SUNY Buffalo

1. From Proposition to Predicate

From Proposition to Predicate

$3 + 2 = 5$ is a proposition. But is $X + 2 = 5$ a proposition?

- Because it has a variable X in it, we cannot say it is T or F.
- So, it is not a proposition. It is called a predicate.

From Proposition to Predicate

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- So, it is not a proposition. It is called a predicate.

Definition

A predicate is a function. It takes some variable(s) as arguments; it returns either True or False (but not both) for each combination of the argument values.

- In contrast, a proposition is not a function. It does not have any variable as argument. It is either True or False (but not both).
- The variables are always associated with a universe (or domain) of discourse, which tells us what combinations of the argument values are allowed.

Predicate vs. Proposition

- Suppose $P(x)$ is a predicate, where the universe of discourse for x is $\{1, 2, 3\}$. Then $P(x)$ is not a proposition, but $P(1)$ is a proposition.
- In general, a predicate is not a proposition. But when you assign values to all its argument variables, you get a proposition.

Predicate vs. Proposition

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- In general, a predicate is not a proposition. But when you assign values to all its argument variables, you get a proposition.

Example:

$P(x, y):$ "$x + 2 = y$" is a predicate.

- It has two variables x and y;
- Universe of Discourse: x is in $\{1, 2, 3\}$; y is in $\{4, 5, 6\}$.

- $P(1, 4):$ $1 + 2 = 4$ is a proposition (it is F);
- $P(2, 4):$ $2 + 2 = 4$ is a proposition (it is T);
- $P(2, 3):$ meaningless (in this example), because 3 is not in the specified universe of discourse for y.

Example

Example

Suppose $Q(x, y) :$ "$x + y > 4$", where the universe of discourse is all integer pairs.

Which of the following are predicates? Which are propositions?

- $Q(1, 2)$
- $Q(1000, 2)$
- $Q(x, 2)$
- $Q(1000, y)$
- $Q(x, y)$

Universal quantification

Universal quantification is another way of converting a predicate into a proposition.

Definition

Suppose $P(x)$ is a predicate on some universe of discourse.

- The universal quantification of $P(x)$ is the proposition:
  "$P(x)$ is true for all x in the universe of discourse".

- We write $\forall x P(x)$, and say "for all x, $P(x)$".

- $\forall x P(x)$ is TRUE if $P(x)$ is true for every single x.

- $\forall x P(x)$ is FALSE if there is an x for which $P(x)$ is false.

Universal quantification example

Example:

$P(x)$:   "$x + 2 = 5$, universe of discourse: $\{1, 2, 3\}$.

$\forall\, x\, P(x)$ means: "for all x in $\{1, 2, 3\}$, $x + 2 = 5$".
In other words, it means: "1+2=5, 2+2=5, and 3+2=5", which is false.
So it is a proposition.

More example:

$A(x)$:   "$x = 1$" $B(x)$:   "$x > 2$" $C(x)$:   "$x < 2$"
Universe of discourse is $\{1, 2, 3\}$.

True or False?
a) $\forall\, x\, (C(x) \to A(x))$
b) $\forall\, x\, (C(x) \lor B(x))$

Existential quantification

Existential quantification is yet another way of converting a predicate into a proposition.

Definition

Suppose $P(x)$ is a predicate on some universe of discourse.

- The existential quantification of $P(x)$ is the proposition:
  "$P(x)$ is true for some x in the universe of discourse."
- We write $\exists\, x\, P(x)$, and say "there exists x, $P(x)$".
- $\exists\, x\, P(x)$ is TRUE if $P(x)$ is true for any x.
- $\exists\, x\, P(x)$ is FALSE if for every x, $P(x)$ is false.

Existential quantification example

Example:

$P(x)$: "$x + 2 = 5$", universe of discourse: $\{1, 2, 3\}$.

$\exists\, x\, P(x)$ means: "for some x in $\{1, 2, 3\}$, $x + 2 = 5$".
In other words, it means: "$1 + 2 = 5$, or $2 + 2 = 5$, or $3 + 2 = 5$" which is true. So it is a proposition.

More examples on existential quantifier:

$A(x)$: "$x = 1$", $B(x)$: "$x > 5$", $C(x)$: "$x < 5$".
Universe of discourse is $\{1, 2, 3\}$.

True or False?
a) $\exists\, x\, (C(x) \rightarrow A(x))$
b) $\exists\, x\, B(x)$

More examples

Example:

$A(x)$: "x lives in Amherst."
$B(x)$: "x is a CSE 191 student."
$C(x)$: "x has a good GPA."
$D(x)$: "x majors in computer science."
Universe of discourse: all UB students.

- All CSE 191 students have good GPA: $\forall x \ (B(x) \rightarrow C(x))$
- No CSE 191 student lives in Amherst: $\neg \exists x \ (B(x) \wedge A(x))$
- CSE 191 students who do not live in Amherst major in computer science: $\forall x \ (B(x) \wedge \neg A(x) \rightarrow D(x))$

Quantifier negation

Example:

Consider the following two propositions:

- Not every UB student majors in computer science: $\neg \, \forall x \, D(x)$.
- There is UB student who does not major in computer science: $\exists \, x \, \neg \, D(x)$.

Do these two statements have the same meaning? YES.

Example:

Consider the following two propositions:

- There is no UB student living in Amherst: $\neg \, \exists \, x \, A(x)$.
- Every UB student lives in a town other than Amherst: $\forall \, x \, \neg \, A(x)$.

Do these two statements have the same meaning? YES.

Quantifier negation

Quantifier negation: In general we have:

$$\neg \, \forall \, x \, P(x) \;\; \equiv \;\; \exists \, x \, \neg \, P(x)$$
$$\neg \, \exists \, x \, P(x) \;\; \equiv \;\; \forall \, x \, \neg \, P(x)$$

Where $\equiv$ means logical equivalence as we have seen.

Rule: to negate a quantifier:
- move the negation to the inside;
- and switch $\exists$ to $\forall$, $\forall$ to $\exists$.

Nested Quantifier

- Sometimes we have more complicated logical formulas that require nested quantifiers.
- "Nested" means one quantifier's scope contains another quantifier.

- $\forall x \forall y P(x, y) = T$ if $P(x, y) = T$ for all x and y.
- $\forall x \exists y P(x, y) = T$ if for any x there exists y such that $P(x, y) = T$.
- $\exists x \forall y P(x, y) = T$ if there exists x such that $P(x, y) = T$ for all y.
- $\exists x \exists y P(x, y) = T$ if these exists x and there exists y such that $P(x, y) = T$.

Examples for translating logic formulas to sentences

Example:

Suppose $P(x, y)$ means: "x (the first variable in $P(*.*)$ loves y (the second variable in $(P(*, *)$"), where the domain of x is the students in this class and the domain of y is the courses offered by UB CSE.

Note: The variable is defined by its position in $P(*, *)$. We might use different names for them. So when we write $P(y, w)$, we are saying y loves w.

- $P(Alice, CSE191)$: Alice loves CSE191.
- $\exists y \, P(Alice, y)$: Alice loves a UB CSE course.
- $\exists x \, (P(x, CSE191) \wedge P(x, CSE250))$:
  A student in this class loves both CSE 191 and CSE 250.
- $\exists x \, \exists y \, \forall z \, ((x \neq y) \wedge (P(x, z) \rightarrow P(y, z)))$:
  There are two different students x and y in this class such that if x loves a UB CSE course, then y loves it as well.

Example for translating sentences to logic formulas

Now we consider translations in the other direction.

Example:

Suppose $P(x, y)$ means: "x loves y", where the domain of x is the students in this class and the domain of y is the courses offered by UB CSE.

- Every UB CSE course is loved by some student in this class:
  $\forall y \exists x \, P(x, y)$.
- No student in this class loves both CSE 191 and CSE 250:
  $\neg \exists x \, (P(x, CSE191) \wedge P(x, CSE250))$.

Nested quantifier: examples

Example:

$A(x)$: "x lives in Amherst."
$C(x, y)$: "x and y are friends."
$D(x)$: "x majors in computer science."
Universe of discourse: all UB students.

There is a computer science major who has a friend living in Amherst:
$\exists x\ (D(x) \wedge \exists y\ (A(y) \wedge C(x, y)))$

Example:

$A(x)$: "x lives in Amherst."
$B(x)$: "x is a CSE 191 student."
$C(x, y)$: "x and y are friends."
$D(x)$: "x majors in computer science."
Universe of discourse: all UB students.

Either there is a computer science major who has a friend living in Amherst, or all CSE 191 students major in computer science:

$\exists x\ (D(x) \wedge \exists y\ (A(y) \wedge C(x, y))) \vee \forall x\ (B(x) \rightarrow D(x))$

Compare examples

Example:

$A(x)$: "x lives in Amherst."
$B(x)$: "x is a CSE 191 student."
$C(x, y)$: "x and y are friends."
$D(x)$: "x majors in computer science."
Universe of discourse: all UB students.

Consider the following two statements:

- $\forall x \, \exists y (C(x, y) \wedge B(y))$:
  All UB students have friends taking CSE191.

- $\exists y \, \forall x \, (C(x, y) \wedge B(y))$:
  There is a UB student who is the friend of all UB students and takes CSE 191.

Do they mean the same thing? NO.

Order Matters

Difference in orders of quantifiers may lead to difference in truth values.

Example:

Let the universe of discourse be pairs of real numbers.

- $\forall x \exists y \, (y > x) \equiv$ True
  But
- $\exists y \forall x \, (y > x) \equiv$ False

In general:

- $\forall x \forall y P(x, y) = T$ iff $\forall y \forall x P(x, y) = T$.
- $\exists x \exists y P(x, y) = T$ iff $\exists y \exists x P(x, y) = T$.
- If $\exists x \forall y P(x, y) = T$, then $\forall y \exists x P(x, y) = T$.
- If $\forall y \exists x P(x, y) = T$, $\exists x \forall y P(x, y)$ might be F.

Example:

$A(x)$: "x lives in Amherst."
$B(x)$: "x is a CSE 191 student."
$C(x, y)$: "x and y are friends."
$D(x)$: "x majors in computer science."
Universe of discourse: all UB students.

- $\exists\, x\, D(x) \,\wedge\, \exists\, x\, \exists\, y\, (A(y) \wedge C(x, y))$:
  There is a computer science major and there is a UB student who has a friend living in Amherst.
  Compared with:
- $\exists\, x\, (D(x) \wedge \exists\, y\, (A(y) \wedge C(x, y)))$:
  There is a computer science major who has a friend living in Amherst.

The meaning of the two statements are different. This is because the scopes of the variable $\exists\, x$ in the two formula are different.

Scope matters

In general, difference in scopes of quantifiers can lead to difference in truth values.

Example:

- $\forall\, x\; (P(x) \lor \neg\, P(x)) \equiv$ True
  But the truth value of

- $\forall\, x\; P(x) \lor \neg\, P(x)$
  depends on the truth value of free variable x. (The second x is not bounded by any quantifier.) So this formula is not even a proposition!

Example:

$A(x)$: "x lives in Amherst."
$B(x)$: "x is a CSE 191 student."
$C(x)$: "x has a good GPA."
$D(x)$: "x majors in computer science."
Universe of discourse: all UB students.

If all computer science majors have friends taking CSE191, then there are a pair of friends both taking CSE191.

$\forall x \, (D(x) \rightarrow \exists y \, (B(y) \wedge C(x, y))) \rightarrow \exists x \, \exists y \, (B(x) \wedge B(y) \wedge C(x, y))$

Rules for quantifier scope

We have the following important rules:

$$\forall x \, (P(x) \wedge Q(y)) \equiv \forall x \, P(x) \wedge Q(y)$$
$$\forall x \, (P(x) \vee Q(y)) \equiv \forall x \, P(x) \vee Q(y)$$
$$\exists x \, (P(x) \wedge Q(y)) \equiv \exists x \, P(x) \wedge Q(y)$$
$$\exists x \, (P(x) \vee Q(y)) \equiv \exists x \, P(x) \vee Q(y)$$

In general, as long as a subformula does not contain a quantifier's bound variable, you can arbitrarily put it into the scope of that quantifier, or take it out of the scope of that quantifier.