Sets and Set Operations

Class Note 04:
Sets and Set Operations
Computer Sci & Eng Dept
SUNY Buffalo

Sets

Definition:

A Set is a collection of objects that do NOT have an order.

- Each object is called an element.
- We write $e \in S$ if e is an element of S; and $e \notin S$ if e is not an element of S.

Set is a very basic concept used in all branches of mathematics and computer science.

How to describe a set:

- Either we list all elements in it, e.g., $\{1, 2, 3\}$.
- Or we specify what kind of elements are in it, e.g., $\{a \mid a > 2, a \in R\}$.
  (Here R denotes the set of all real numbers).

Example sets

- $N = \{0, 1, 2, \ldots\}$: the set of natural numbers.
  (Note: in some books, 0 is not considered a member of N.)
- $Z = \{0, -1, 1, -2, 2, \ldots\}$: the set of integers.
- $Z^+ = \{1, 2, 3, \ldots\}$: the set of positive integers.
- $Q = \{p/q \mid p \in Z,\ q \in Z,\ q \neq 0\}$: the set of rational numbers.
- $Q^+ = \{x \mid x \in Q,\ x > 0\}$: the set of positive rational numbers.
- R: the set of real numbers.
- $R^+ = \{x \mid x \in R,\ x > 0\}$: the set of positive real numbers.

Definition:
The empty set, denoted by $\emptyset$, is the set that contains no elements.

More example sets

- A={Orange, Apple, Banana} is a set containing the names of three fruits.
- B={Red, Blue, Black, White, Grey} is a set containing five colors.
- $\{x \mid x$ takes CSE191 at UB in Spring 2014$\}$ is a set of 220 students.
- {N,Z,Q,R} is a set containing four sets.
- $\{x \mid x \in \{1, 2, 3\}$ and $x > 1\}$ is a set of two numbers.

Note: When discussing sets, there is a universal set U involved, which contains all objects under consideration. For example: for A, the universal set might be the set of names of all fruits. for B, the universal set might be the set of all colors.
In many cases, the universal set is implicit and omitted from discussion. In some cases, we have to make the universal set explicit.

Equal sets

Definition:
Two sets are equal if and only if they have the same elements.

- Note that the order of elements is not a concern since sets do not specify orders of elements.
- We write $A = B$, if A and B are equal sets.

Example:
- {1,2,3} = {2,1,3}
- $\{1, 2, 3, 4\} = \{x \in Z$ and $1 \leq x < 5\}$

Subset

Definition:
A set A is a subset of B if every element of A is also in B.

- We write $A \subseteq B$ if A is a subset of B.
- Clearly, for any set A, the empty set $\emptyset$ (which does not contain any element) and A itself are both subsets of A.

Definition:
If $A \subseteq B$ but $A \neq B$, then A is a proper subset of B, and we write $A \subset B$.

Fact:
Suppose A and B are sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

This fact is often used to prove set identities.

Cardinality

Definition:

If a set A contains exactly n elements where n is a non-negative integer, then A is a finite set, and n is called the cardinality of A. We write $|A| = n$.

- For a finite set, its cardinality is just the "size" of A.
- Note: $\emptyset$ is the empty set (containing no element); $\{\emptyset\}$ is the set containing one element (which is the empty set).

Example:

- $|\{x \mid -2 < x < 5, x \in Z\}| = ?$
- $|\emptyset| = ?$
- $|\{x \mid x \in \emptyset$ and $x < 3\}| = ?$
- $|\{x \mid x \in \{\emptyset\}\}| = ?$

Cardinality of infinite set

Definition:

If A is not finite, then it is an infinite set.

- What is the cardinality (i.e. the size) of an infinite set?
- Do all infinite sets have the same size (i.e ∞)?
- Apparently, they do not: It appears that there are more rational numbers than integers and there are more real numbers than rational numbers. (I say appears because, with proper definition, only one of these two statements is true.)
- But how do we define the notion: "an infinite set contains more elements than another infinite set"?
- We shall deal with this later.

Power set

Definition

The power set of set A is the set of all subsets of A. We denote it by $P(A)$.

Example:

- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
- $P(\emptyset) = \{\emptyset\}$.
- $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

Fact:

In general, $|P(A)| = 2^{|A|}$.

Ordered tuple

- Recall that a set does not consider its elements order.
- But sometimes, we need to consider a sequence of elements, where the order is important.
- An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ has a_1 as its first element, a_2 as its second element, . . ., a_n as its n^{th} element.
- The order of elements is important in such a tuple.
- Note that $(a_1, a_2) \neq (a_2, a_1)$ but $\{a_1, a_2\} = \{a_2, a_1\}$.

Cartesian product

Definition:

The Cartesian product of $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is defined as the set of ordered tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$. That is:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Example Cartesian products

Examples:

- $\{1,2\} \times \{3,4,5\} = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$.
- {Male, Female} × {Married, Single} × {Student, Faculty} = {(Male, Married, Student), (Male, Married, Faculty), (Male, Single, Student), (Male, Single, Faculty), (Female, Married, Student), (Female, Married, Faculty), (Female, Single, Student), (Female, Single, Faculty)}.
- $R \times R = \{(x,y) \mid x \in R, y \in R\}$ is the set of point coordinates in the 2D plane.

Cardinality of Cartesian product

Fact:

In general, if A_i's are finite sets, we have:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

Example:

$|\{(s, g) \mid s$ is a CSE191 student and g is a letter grade$\}| = ?$

Solution:

- $|\{$ CSE 191 student $\}| = 220$;
- $|\{$ Letter grade $\}| = 10$;
- So, $|\{(s, g)|s$ is a CSE 191 student and g is a letter grade $\}|$ = 220 x 10 = 2200.

Using set notation with quantifiers

Sometimes, we restrict the domain of a quantified statement explicitly by using set notations.

- We use $\forall\, x \in S\ (P(x))$ to denote that $P(x)$ holds for every $x \in S$.
- We use $\exists\, x \in S(P(x))$ to denote that $P(x)$ holds for some $x \in S$.

Example:

$\forall\, x \in R\ (x^2 \geq 0)$ means that the square of every real number is greater than or equal to 0.

Truth set

Definition:

Consider a domain D and a predicate $P(x)$. The truth set of P is the set of elements x in D for which $P(x)$ holds.

- Using the notation of sets, we can write $\{x \in D \mid P(x)\}$.
- Clearly, it is a subset of D. It is equal to D if and only if $P(x)$ holds for all $x \in D$.

Example:

- $\{x \in \{1, 2, 3\} \mid x > 1.5\} = \{2, 3\}$.
- $\{x \in R \mid x^2 = 0\} = \{0\}$.
- $\{x \in R \mid x^2 \geq 0\} = R$.

Set operations

Recall:

- We have $+, -, \times, \div, \ldots$ operators for numbers.
- We have $\vee, \wedge, \neg, \rightarrow \ldots$ operators for propositions.

Question:

What kind of operations do we have for sets?

Answer: union, intersection, difference, complement, ...

Set Union

Definition:

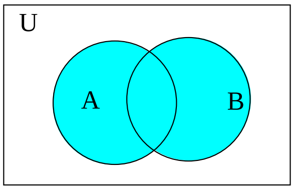
The union of two sets A and B is the set that contains exactly all the elements that are in either A or B (or in both).

- We write $A \cup B$.
- Formally, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Example:

- $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$
- $\{x \mid x > 0\} \cup \{x \mid x > 1\} = \{x \mid x > 0\}$

Venn Diagram of Union Operation:



Set intersection

Definition:

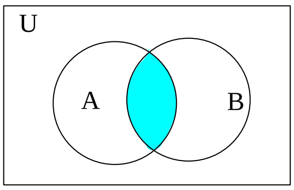
The intersection of two sets A and B is the set that contains exactly all the elements that are in both A and B.

- We write $A \cap B$.
- Formally, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Example:

- $\{1,2,3\} \cap \{3,4,5\} = \{3\}$
- $\{x \mid x > 0\} \cap \{x \mid x > 1\} = \{x \mid x > 1\}$

Venn Diagram of Intersection Operation:



Disjoint set

Definition:

Two sets A and B are disjoint if $A \cap B = \emptyset$.

Example:

- $\{1, 2, 3\} \cap \{4, 5\} = \emptyset$, so they are disjoint.
- $\{1, 2, 3\} \cap \{3, 4, 5\} \neq \emptyset$, so they are not disjoint.
- $Q \cap R^+ \neq \emptyset$, so they are not disjoint.
- $\{x \mid x < -2\} \cap R^+ = \emptyset$, so they are disjoint.

Cardinality of intersection and union

Lemma:

For any two sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Intuitively, when we count the elements in A and the elements in B separately, those elements in $A \cap B$ have been counted twice. So when we subtract $|A \cap B|$ from $|A| + |B|$, we get the cardinality of the union.
- An extension of this result is called the inclusion-exclusion principle. We will discuss this later.

Set complement

Definition:

The complement of set A, denoted by $\overline{A}$, is the set that contains exactly all the elements that are not in A.

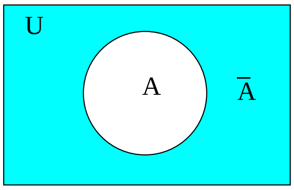
- Formally, $\overline{A} = \{x \mid x \notin A\}$.
- Suppose U is the universe. Then, $\overline{A} = U - A$.

Example:

Let the universe be R.

- $\overline{\{0\}} = \{x \mid x \neq 0 \wedge x \in R\}$ $\overline{R^+} = \{x \mid x \leq 0 \wedge x \in R\}$.

Venn Diagram of Complement Operation:



Set difference

Definition:

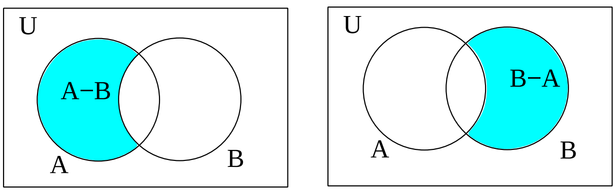
The difference of set A and set B, denoted by $A - B$, is the set that contains exactly all elements in A but not in B.

- Formally, $A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$.

Example:

- $\{1,2,3\} - \{3,4,5\} = \{1,2\}$
- $R - \{0\} = \{x \mid x \in R \wedge x \neq 0\}$
- $Z - \{2/3, 1/4, 5/8\} = Z$

Venn Diagram of Difference Operation:



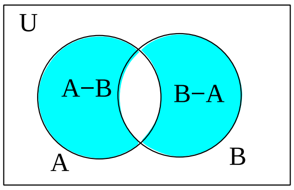
Symmetric Difference

Definition:

The symmetric difference of set A and set B, denoted by $A \oplus B$, is the set containing those elements in exactly one of A and B.

- Formally: $A \oplus B = (A - B) \cup (B - A)$.

Venn Diagram of Symmetric Difference Operation:



Example for set operations

Suppose A is the set of students who loves CSE 191, and B is the set of students who live in the university dorm.

- $A \cap B$: the set of students who love CSE 191 and live in the university dorm.
- $A \cup B$: the set of students who love CSE 191 or live in the university dorm.
- $A - B$: the set of students who love CSE 191 but do not live in the university dorm.
- $B - A$: the set of students who live in the university dorm but do not love CSE 191.

Example for calculating set operations

Example:

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then:

- $A \cap B = \{1, 2, 3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A - B = \emptyset$
- $B - A = \{6, 7, 8\}$

Representing Sets in Computer Programs

Set is an important data structure in CS. How to represent sets in computer programs?

- Let $U = \{s_1, s_2, \ldots, s_n\}$ be the universal set.
- We can use an array S of n-bits to represent the sets in U and the set operations. Let $S[i]$ be the ith bit in S. Each $S[i]$ is either 0 or 1.
- To represent a subset $A \subseteq S$, we use:

$$S_A[i] = \begin{cases} 0 & \text{if } s_i \notin A \\ 1 & \text{if } s_i \in A \end{cases}$$

This is called the bit map representation of sets (discussed in CSE250).

Example of bit map representations

Example: $U = \{a, b, c, d, e, f, g\}$.

- $A = \emptyset$: $S_A = [0, 0, 0, 0, 0, 0, 0]$.
- $A = U$: $S_A = [1, 1, 1, 1, 1, 1, 1]$.
- $A = \{b, d, f\}$: $S_A = [0, 1, 0, 1, 0, 0, 1]$.

Logic Operators in Java and C++

Logic Operators in Java and C++

- Bitwise and: &

$$\begin{array}{rrrrrrr} & 0 & 1 & 0 & 1 & 0 & 1 \\ \& & 1 & 1 & 1 & 0 & 0 & 1 \\ \hline & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

- Bitwise or: |

$$\begin{array}{rrrrrrr} & 0 & 1 & 0 & 1 & 0 & 1 \\ | & 1 & 1 & 1 & 0 & 0 & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 & 1 \end{array}$$

- Bitwise exclusive or: ˆ

$$\begin{array}{rrrrrrr} & 0 & 1 & 0 & 1 & 0 & 1 \\ \hat{} & 1 & 1 & 1 & 0 & 0 & 1 \\ \hline & 1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

Using C++ operators to calculate set operations

Using C++ operators to calculate set operations:

Let A and B be two subsets of U. Let S_A and S_B be the bit map representation of A and B, respectively.

- $S_{A\cap B} = S_A \ \& \ S_B$;
- $S_{A\cup B} = S_A \mid S_B$;
- $S_{A\oplus B} = S_A \hat{} S_B$;
- $S_{\overline{A}} = S_A \hat{} [1, 1, \ldots, 1]$;

Set Identities: distributivity

Set operations satisfy several laws. If we consider:

- $\cap$ similar to $\wedge$;
- $\cup$ similar to $\vee$;
- $\overline{A}$ similar to $\neg A$;
- The universal set U similar to T;
- The empty set $\emptyset$ similar to F;

then, they are very similar to the logic laws.

Set Identities: distributivity

Distributivity laws:

Just like the distributivity in logical equivalence, for sets we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

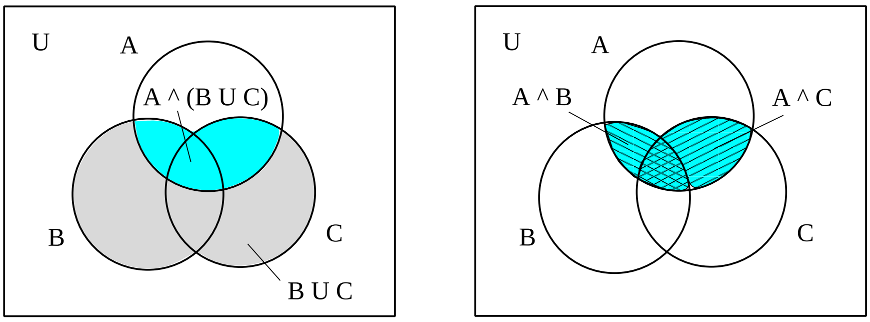
Example:

{red, blue} ∩ ({red, black, white} ∪ {blue})
= {red, blue} ∩ {red, black, white, blue}
= {red,blue}
= {red} ∪ {blue}
= ({red,blue} ∩ {red,black,white}) ∪ ({red,blue} ∩ {blue})

Set Identities: distributivity

Venn Diagram Proof for:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



Set identities: DeMorgan law

DeMorgan laws:

Just like the DeMorgan laws in logic, we have:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Example:

Let the universe be $\{0,1,2,3\}$.
$\overline{\{0,1\} \cap \{1,2\}} = \overline{\{1\}} = \{0,2,3\} = \{2,3\} \cup \{0,3\} = \overline{\{0,1\}} \cup \overline{\{1,2\}}$.

Other identities

Table: Set Identifies

Identify	Name
$A \cap U = A$ $A \cup \emptyset = A$	**Identity laws**
$A \cup U = U$ $A \cap \emptyset = \emptyset$	**Domination laws**
$A \cup A = A$ $A \cap A = A$	**Idempotent laws**
$\overline{(\overline{A})} = A$	**Complementation law**
$A \cup B = B \cup A$ $A \cap B = B \cap A$	**Commutative laws**
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	**Associative laws**
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	**Distributive laws**
$\overline{A \cup B)} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	**De Morgan's laws**
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	**Absorption laws**
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	**Complement laws**

You should get familiar with these laws, so that you can use them to prove set identities.

How to prove set identities

To prove a set identity:

$$X = Y$$

it is necessary and sufficient to show two things:

- $X \subseteq Y$ and
- $Y \subseteq X$

Equivalently, it is necessary and sufficient to show two things:

- $x \in X \rightarrow x \in Y$ and
- $x \in Y \rightarrow x \in X$

Example for proving set identities

Example:

Show that if A, B, C are sets, then:

$$(\overline{A} - B) - C = (\overline{A} - C) - (B - C)$$

Proof: First we show $x \in$ LHS $\rightarrow x \in$ RHS:

$x \in (\overline{A} - B) - C$ (by the definition of "set difference")
$\Rightarrow x \in (\overline{A} - B)$ but $x \notin C$. Hence $x \in \overline{A}, x \notin B$ and $x \notin C$.
So $x \in \overline{A} - C$ and $x \notin B - C$. This means $x \in (\overline{A} - C) - (B - C) =$ RHS.

Next we show $x \in$ RHS $\rightarrow x \in$ LHS:

$x \in (\overline{A} - C) - (B - C)$ (by definition of "set difference")
$\Rightarrow x \in (\overline{A} - C)$ but $x \notin (B - C)$. Hence: $x \in \overline{A},\ x \notin C$ and $x \notin (B - C)$.
Here: $x \notin B - C$ means either $x \notin B$ or $x \in C$.
Since the latter contradicts $x \notin C$, we must have $x \notin B$.
This implies $x \in (\overline{A} - B) - C =$ LHS.

Prove set identities by truth table

We can also prove the identify by using membership table (which is similar to truth table):

A	B	C	$\overline{A}$	$\overline{A}-B$	$\overline{A}-C$	$B-C$	$(\overline{A}-B)-C$	$(\overline{A}-C)-(B-C)$
T	T	T	F	F	F	F	F	F
T	T	F	F	F	F	T	F	F
T	F	T	F	F	F	F	F	F
T	F	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F	F
F	T	F	T	F	T	T	F	F
F	F	T	T	T	F	F	F	F
F	F	F	T	T	T	F	T	T

- Each row specifies membership conditions. For example, the row 1 is $\{x \mid x \in A, x \in B, x \in C\}$; the row 2 is $\{x \mid x \in A, x \in B, x \notin C\}$.
- The last two columns are identical. So the two sets are the same.

Generalized Union

- The previously studied union operation applies to only two sets.
- We can generalize it to n sets.
- Generally speaking, the union of a collection of sets is the set that contains exactly those elements that are in at least one of the sets in the collection.
- We write:

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- That is:

$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_1 \vee x \in A_2 \vee \ldots \vee x \in A_n\}$$

Example for generalized union

Example 1:

Suppose that $A_i = \{1, 2, \ldots, i\}$ for all positive integer i. Then

$$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\} = A_n$$

Example 2:

Suppose that $A_i = \{i+1, i+2, \ldots, 2i\}$ for all positive integer i. Then:

$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= \{2\} \cup \{3,4\} \cup \{4,5,6\} \cup \ldots \cup \{n+1, n+2, \ldots, 2n\} \\ &= \{2, 3, 4, \ldots, 2n\} = \{x \mid 2 \leq x \leq 2n, x \in Z\} \end{aligned}$$

Generalized intersection

- Similarly, we can generalize intersection to n sets.
- Generally speaking, the intersection of a collection of sets is the set that contains exactly those elements that are in all of the sets in the collection.
- We write:

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

- That is:

$$\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_1 \wedge x \in A_2 \wedge \ldots \wedge x \in A_n\}$$

Example for generalized intersection

Example 1:

Suppose that $A_i = \{1, 2, \ldots, i\}$ for all positive integer i. Then:

$$\bigcap_{i=1}^{n} A_i = \{1\} = A_1$$

Example 2:

Suppose that $A_i = \{i+1, i+2, \ldots, 2i\}$ for all positive integer i. Then:

$$\begin{aligned} \bigcap_{i=1}^{n} A_i &= \{2\} \cap \{3, 4\} \cap \{4, 5, 6\} \cap \ldots \cap \{n+1, n+2, \ldots, 2n\} \\ &= \emptyset \end{aligned}$$