CSE 191, Class Note 05: Counting Methods

Computer Sci & Eng Dept
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Need for Counting

The problem of counting the number of different objects appears in many computer science applications. Some examples:

- All Java programs use identifiers. Say, each identifier is limited to 8 characters long, the first character must be a letter. How many possible identifiers are there?
- In GPS map routing application, we know the starting location, the ending location, and the intermediate locations on the map, how many different routes are there? (The routing program must select the best route from these routes.)
- In Facebook application, we are given three users A, B, C. We know the set of the friends of A, B, C. How do we count the number of users who are the friend of at least one of A, B, C?

We will discuss some commonly used counting methods.

Outline

Product Rule

Product Rule

Suppose that a procedure can be broken into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

Example 1:

The chairs in an auditorium are labeled by upper case English letters followed by an integer between 0 and 99. What's the largest numbers of chairs that can be labeled this way?

Solution: There are 26 ways to pick letters. There are 100 ways to pick integers. So, by Product Rule, there are $26 \cdot 100 = 2600$ ways to label chairs.

Product Rule

The Product Rule can be easily generalized from 2 tasks to k tasks for any integer k.

Example 2:

How many different bit strings of length 8 are there?

Solution: $2^8 = 256$.

Example 3:

Suppose that Java program identifiers consist of only upper and lower cases English letters and digits. The first character must be an upper case letter. The last character must be a digit. (They are not real requirements, just an example). Count the number of possible identifiers of length exactly 6.

Solution: There are 26 choices for the 1st character. There are 26+26+10=62 choices for each of the 2nd - 5th character. There are 10 choices for the last character. So the answer is $26 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 10$.

Product Rule

Notation: $|A|$ denotes the number of elements in the set A.

Product Rule: $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|$.

Example 4:

Show that the number of different subsets of a finite set S is $2^{|S|}$.

Let $S = \{s_1, s_2, \ldots s_n\}$ be a finite set containing $|S| = n$ elements. To describe a subset $A \subseteq S$, we need to make the following choices:

- whether to include s_1 or not; (2 choices);
- whether to include s_2 or not; (2 choices);
- $\ldots$
- whether to include s_n or not; (2 choices);

The different choices in each step (n steps in total) will result in different subsets. So the number of different subsets of S is $2^n = 2^{|S|}$.

Outline

Sum Rule

Sum Rule

Let $A_1, A_2, \ldots, A_m$ be different sets such that $A_i \cap A_j = \emptyset$ for all $i \neq j$. Then:

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

Example 5:

In a high school, 32 students take French class, 45 students take Spanish. Assuming no student take both French and Spanish, how many students take either French or Spanish?

Solution: Simple enough: 32+45= 77.

It is simple because we assumed $A_i \cap A_j = \emptyset$. What if we drop this assumption?

Inclusion-Exclusion Principle (with two sets)

Let A_1, A_2 be two sets. Then:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example 5a:

In a high school; 32 students take French class, 45 students take Spanish. 15 student take both French and Spanish. How many students take either French or Spanish?

Solution: $32 + 45 - 15 = 62$.

Inclusion-Exclusion Principle

Example 7:

Determine the number of 8-bits binary strings such that:

- Either the first bit is 1;
- Or the last two bits are 00.

Let A be the set of 8-bits binary strings whose first bit is 1.
Let B be the set of 8-bits binary strings whose last two bits are 00.
We want to determine $|A \cup B|$.

- $|A| = 2^7$ (because the first bit is fixed.)
- $|B| = 2^6$ (because the last two bits are fixed.)
- $|A \cap B| = 2^5$ (because the first and the last two bits are fixed.)

So $|A \cup B| = 2^7 + 2^6 - 2^5 = 160$.

This principle can be generalized to more than two sets.

Inclusion-Exclusion Principle

Inclusion-Exclusion Principle (with three sets)

Let A_1, A_2, A_3 be three sets. Then:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$$

Example 6:

In a high school, 32 students take French class, 45 students take Spanish, 20 take German. 15 students take both French and Spanish; 12 students take both German and Spanish; 10 students take both French and German; 6 students take all three foreign languages. How many students take at least one foreign language?

Solution: $32 + 45 + 20 - (15 + 12 + 10) + 6 = 66$.

Inclusion-Exclusion Principle

Inclusion-Exclusion Principle (general case):

Let $A_1, A_2, \ldots A_m$ be m sets. Then:

$$\begin{aligned}
|A_1 \cup \ldots \cup A_m| \quad = \quad & |A_1| + |A_2| + \ldots + |A_m| \\
& -(\text{sum of } |A_i \cap A_j| \text{ for all possible } i \neq j) \\
& +(\text{sum of } |A_i \cap A_j \cap A_k| \text{ for all possible } i \neq j \neq k) \\
& -(\text{sum of } |A_i \cap A_j \cap A_k \cap A_l| \text{ for all possible } i \neq j \neq k \neq l) \\
& \ldots \\
& +(-1)^{m-1} |A_1 \cap A_2 \cap \ldots \cap |A_m|
\end{aligned}$$

Exercise:

Write the general formula for the case $m = 4$.

Note:

For $m > 4$, this formula is too complicated to be useful.

Outline

Permutations

Definition

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an r-permutation.
- The number of r-permutations of a set with n elements is denoted by $P(n, r)$.

Permutations

Example 1:

In how many ways can we select 3 students from a group of 5 students to stand in line for a picture?

Solution:

- We can select any of the 5 students for the first position.
- After the first position is selected, we can choose any of the 4 remaining students for the 2nd position.
- After the 1st and the 2nd positions are selected, we can choose any of the 3 remaining students for the 3rd position.

  So, the answer is: $5 \cdot 4 \cdot 3 = 60$.

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In this example, we are counting the number of distinct 3-permutations in a set of 5 elements. What we have shown is:

$$P(5,3) = 5 \cdot 4 \cdot 3 = 60$$

Permutations

Definition:

The factorial of n is denoted by $n!$ and defined by:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

Note: For convenience, we define: 0!=1.

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Theorem 1:

$$\begin{aligned} P(n,r) &= \underbrace{n \cdot (n-1) \cdots (n-r+1)}_{r \text{ terms}} = \frac{n!}{(n-r)!} \\ P(n,n) &= n \cdot (n-1) \cdots 2 \cdot 1 = n! \end{aligned}$$

Example 2:

How many ways are there to select a first-prize winner; a second-prize winner; and a third-prize winner from 100 different people?

Solution: $P(100,3) = 100 \cdot 99 \cdot 98 = 970200$.

Permutations

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Example 3:

A salesman must visit eight different cities. He must begin his trip from a specific city (his home), but he can visit other seven cities in any order, then he must return to his home city. How many possible orders can he use?

Solution: $7! = 5040$.

Example 3:

How many permutations of the letters in $\{A, B, C, D, E, F, G\}$ contain the string ABC.

Solution: There are 7 letters. Is $7!$ the solution?

No. Since the string ABC must appear as a block, it is like a single symbol. So we are really asking the number of permutations from 5 symbols. So the solution is $5! = 120$.

Outline

Combinations

Definition

- A r-combination of elements of a set is an unordered selection of r elements from the set.
- So a r-combination is simply a subset with r elements.
- The number of r-combinations of a set with n elements is denoted by $C(n, r)$ or $\binom{n}{r}$.

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Example 1:

How many different ways to select a committee of three members from 10 students?

Solution: Each committee is just a subset of three elements. So the answer is simply $C(10,3)$.

Combinations

Theorem 2:

For any non-negative integer n and r such that $0 \leq r \leq n$:

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 2 \cdot 1}$$

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$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 2 \cdot 1}$$

Proof: Consider any r-combination, which is just a r element set A. There are $r!$ different orderings of the elements in A. Each of which is an r-permutation of the set. Thus:

$$P(n,r) = C(n,r) \times r!$$

Since $P(n,r) = \frac{n!}{(n-r)!}$, we have:

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \frac{\overbrace{n \cdot (n-1) \cdots (n-r+1)}^{r \text{ terms}}}{r!}$$

Example 1:

How many different ways to select a committee of three members from 10 students?

Solution: $C(10,3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$.

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- (a) How many poker hands of 5 cards can be selected from a standard 52 cards?
- (b) How many hands of 47 cards can be selected from a standard 52 cards?

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Solution:
(a) $C(52,5) = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960.$
(b) $C(52,47) = \frac{52 \cdot 51 \cdot 50 \cdots 7 \cdot 6}{47 \cdot 46 \cdots 3 \cdot 2 \cdot 1}$. Is there a better way?

Combinations

Corollary

$$C(n,r) = C(n,n-r)$$

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Proof: Both of them equal to $\frac{n!}{r!(n-r)!}$.

Combinations

Example 3:

In CSE department, there are 8 full professors, 7 associate professors, and 11 assistant professors. We want to form a committee to evaluate BS degree program. The committee must consist of 4 full professors, 3 associate professors and 3 assistant professors. How many different ways to select the committee?

- The number of ways to select full professors: $C(8,4)$
- The number of ways to select associate professors: $C(7,3)$
- The number of ways to select assistant professors: $C(11,3)$

By product rule, the solution is:

$$C(8,4) \cdot C(7,3) \cdot C(11,3) = \frac{8!}{4!4!} \cdot \frac{7!}{3!4!} \cdot \frac{11!}{3!8!} = 404250$$

Outline

$\binom{n}{r}$ is also called a binomial coefficient. Why?

Example:

$$(x+y)^0 = 1 = \binom{0}{0}$$
$$(x+y)^1 = x+y = \binom{1}{0}x + \binom{1}{1}y$$
$$(x+y)^2 = x^2+2xy+y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$
$\ldots$

$(x+y)^n$ is called a binomial. $\binom{n}{r}$ is a coefficient in the expansion of the binomial.

Binomial Coefficients and Identities

The Binomial Theorem:

Let x and y be variables and n be a non-negative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j =$$

$$\binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

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Proof: The general terms in the product are of the forms $x^{n-j}y^j$ for $0 \leq j \leq n$.

- We need to count the number of the terms of the form $x^{n-j}y^j$.
- To get such a term, we must pick $n-j$ x's (so that the remaining j terms in the product are y's).
- So the number of the terms $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is also $\binom{n}{j}$.

Binomial Identities

Corollary 1: $\sum_{j=0}^{n} \binom{n}{j} = 2^n$.

Proof: Let $x = 1$ and $y = 1$ in the Binomial Theorem, we have:

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j = \sum_{j=0}^{n} \binom{n}{j}$$

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Corollary 2: $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$.

Proof: Let $x = 1$ and $y = -1$ in the Binomial Theorem, we have:

$$0^n = (1-1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (-1)^j = \sum_{j=0}^{n} (-1)^j \binom{n}{j}$$

Pascal Identity

Pascal Identity:

Let n and k be positive integers and $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof: Recall that $\binom{n+1}{k}$ is the # of k-element subsets of a set S with $n+1$ elements. Fix an element $a \in S$. These k-element subsets can be divided into two groups:

1. The subsets that contain a. So we are choosing other $k-1$ elements from $S - \{a\}$. So there are $\binom{n}{k-1}$ such subsets.
2. The subsets that do not contain a. So we are choosing other k elements from $S - \{a\}$. So there are $\binom{n}{k}$ such subsets.

So the sum of the two groups equals to LHS.

Pascal Triangle

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \cdots & & & &
\end{array}
$$

It's values:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \cdots & & & &
\end{array}
$$

Note: The left- and the right-most numbers are always 1. Any other number is the sum of the two numbers "on its shoulder".

Outline

Pigeonhole Principle

The idea of Pigeonhole Principle is very simple. But when used properly, we can get very interesting results.

Pigeonhole Principle (Version 1):

If you put n pigeons into m pigeonholes with $n > m$, then at least one pigeonhole contains at least two pigeons.

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If you put n pigeons into m pigeonholes with $n > m$, then at least one pigeonhole contains at least two pigeons.

Pigeonhole Principle (Version 1, equivalent statement):

If k is a positive integer and $k + 1$ or more objects are put into k boxes, then at least one box contains two or more objects.

Proof: This statement is so obvious that really needs no formal proof. But for completeness, here is a proof by contraposition.

- Suppose that none of k boxes contains more than one object.
- Then the total number of objects in all boxes is at most k.
- This is a contradiction, because there are $k + 1$ objects.

Pigeonhole Principle

Pigeonhole Principle (Version 2):

If you put N pigeons into k pigeonholes, then at least one pigeonhole contains at least $\lceil N/k \rceil$ pigeons.

- $\lceil x \rceil$ denotes the smallest integer that is $\geq x$.
- For example: $\lceil 0.6 \rceil = 1$; $\lceil 3 \rceil = 3$; $\lceil 4.0000001 \rceil = 5$.

Pigeonhole Principle examples

Example 1:

At lease two people in NYC have the same number of hairs.

Pigeonhole Principle examples

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Example 2:

At lease 16 people in NYC have the same number of hairs.

Pigeonhole Principle examples

Example 1:

At lease two people in NYC have the same number of hairs.

Example 2:

At lease 16 people in NYC have the same number of hairs.

- There are 8 million people in NYC.
- It is known that the maximum number of hairs a person can have is 500,000.
- Each person is a "pigeon".
- The "ith pigeonhole" holds the people with i hairs. So there are 500,001 "pigeonholes".
- So at least one pigeonhole contains at least $\lceil 8000000/500001 \rceil = \lceil 15.999968 \rceil = 16$ people.

Example 3:

At any moment, at least 1574 ATT cell phones are connected to a ATT cell phone tower in US.

- There are 40,032 ATT towers, and 63,000,000 ATT cell phones (the numbers are not current, just example)
- So at least $\lceil 63000000/40032 \rceil = 1574$ cell phones are connected to a tower.

Pigeonhole Principle examples

Example 4:

How many cards must be selected from a standard deck of 52 cards to ensure that at least three cards of the same suit are chosen?

Example 5:

How many cards must be selected from a standard deck of 52 cards to ensure that at least three hearts are selected?

Example 6:

If we pick 5 numbers from the integers 1,2,. . ., 8, then two of them must add up to 9.

We will discuss these examples in class.

More difficult examples

Example:

At least two people in this room have the same number of friends in this room. (Of course we must assume there are at least two people in this room.)

Proof: Let k be the number of people who has at least one friend in this room.
Case 1 $k = 0$: Then all people have 0 friends.
Case 2 $k > 0$: Among these k people:

- the numbers of friends they can have are 1, 2,$\ldots, (k-1)$. These choices are $k-1$ pigeonholes.
- But there are k people, each is a "pigeon".
- So at least one pigeonhole contains at least two pigeons. In other words, at least two people have the same number of friends.

More difficult examples

Example:

During a month of 30 days, a baseball team plays at least one game per day. They played exactly 45 games during these 30 days.
Show that there must be a period of consecutive days during which the team must play exactly 14 games.

An equivalent description:

- We have arbitrary 30 integers $a_1, \ldots, a_{30}$.
- Each integer a_i $(1 \leq i \leq 30)$ satisfies $1 \leq a_i$.
- The sum of the 30 integers is 45.
- Show that there is always a continuous section of integers that add up to exactly 14.

We will discuss this problem in class.

More difficult examples

Theorem: Approximate irrational number by rational numbers with small errors:

For any real number a, there exist infinitely many rational numbers p/q (p and q are integers) such that $|a - p/q| < \frac{1}{q^2}$.

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For any real number a, there exist infinitely many rational numbers p/q (p and q are integers) such that $|a - p/q| < \frac{1}{q^2}$.

What we are trying to do?

- Computer can only store finite precision numbers.
- When we store $\pi = 3.1416$, we are saying: $\pi \approx 31416/10000$ with error at most $1/10000$.
- In general, we want to approximate an irrational number a by a rational number p/q with error as small as possible.
- If we insist on that the denominator q must be a power of 10, we can only guarantee the error $\leq \frac{1}{q}$.
- This theorem shows that if we allow arbitrary integer q as the denominator, we can double the accuracy: error $< \frac{1}{q^2}$.

More difficult examples

Example:

$$\begin{aligned} \pi \quad &\approx 3/1 &&= 3 \\ &\approx 22/7 &&= 3.1428571\ldots \\ &\approx 223/71 &&= 3.1408450\ldots \\ &\approx 333/106 &&= 3.1415094\ldots \end{aligned}$$

$\ldots$

Proof: For any number x, let $\{x\}$ denote the fractional part of x, (i.e. the part after the decimal point.) Example:

- $\{0.34\} = 0.34$.
- $\{100.28\} = 0.28$.
- $\{\pi\} = 0.1415926\ldots$.

$[c, d)$ denotes the interval of real numbers x such that $c \leq x < d$

More difficult examples

Let a be the real number we try to approximate by rational numbers. Consider any positive integer Q.

- Consider the fractional parts of the numbers: $\{0\}, \{a \cdot 1\}, \{a \cdot 2\}, \ldots \{a \cdot Q\}$ (these are $Q + 1$ pigeons).
- Consider the Q intervals: $[0, \frac{1}{Q}), [\frac{1}{Q}, \frac{2}{Q}), \ldots, [\frac{Q-1}{Q}, 1)$. (These are the Q pigeonholes.)
- By pigeonhole principle, at least one pigeonhole contains at least two pigeons. In other words, there are q_1 and q_2 $(0 \leq q_1, q_2 \leq Q)$ such that $\{a \cdot q_1\}$ and $\{a \cdot q_2\}$ are in the same interval.
- Then $|\{a \cdot q_1\} - \{a \cdot q_2\}| < \frac{1}{Q}$.
- Suppose: $a \cdot q_1 = x_1.\{a \cdot q_1\}$ and $a \cdot q_2 = x_2.\{a \cdot q_2\}$. (Here x_1 is the integer part of $a \cdot q_1$ and x_2 is the integer part of $a \cdot q_2$.)
- Let $q = q_1 - q_2$ and $p = x_1 - x_2$.

More difficult examples

$$
\begin{aligned}
|q \cdot a - p| &= |(q_1 - q_2) \cdot a - (x_1 - x_2)| \\
&= |q_1 \cdot a - q_2 \cdot a - (x_1 - x_2)| \\
&= |x_1.\{q_1 \cdot a\} - x_2.\{q_2 \cdot a\} - (x_1 - x_2)| \\
&= |(x_1 - x_2) + \{q_1 \cdot a\} - \{q_2 \cdot a\} - (x_1 - x_2)| \\
&= |\{q_1 \cdot a\} - \{q_2 \cdot a\}| \\
&< \frac{1}{Q}
\end{aligned}
$$

This implies: $|a - p/q| < \frac{1}{Q \cdot q} \leq \frac{1}{q^2}$

This completes the proof.

More difficult examples

Example: Let $a = \pi$ and $Q = 8$.

- $0 \cdot \pi = 0$.
- $1 \cdot \pi = 3.14159265$.
- $2 \cdot \pi = 6.28318531$.
- $3 \cdot \pi = 9.42477961$.
- $4 \cdot \pi = 12.56637061$.
- $5 \cdot \pi = 15.70796327$.
- $6 \cdot \pi = 18.84955592$.
- $7 \cdot \pi = 21.99114858$.
- $8 \cdot \pi = 25.13274112$.

Note that $\{1 \cdot \pi\} = 0.14159265$ and $\{8 \cdot \pi\} = 0.13274112$ are in the same interval $[1/8, 2/8)$. So we pick $q_1 = 8, q_2 = 1$. Then $p = 25 - 3 = 22$ and $q = 8 - 1 = 7$. We have $\pi \approx 22/7$.

Outline

Proofs based on parity argument.

Definition

The parity of an integer specifies whether it is even or odd.

Example:

- the parity of 8 and 0 are even.
- the parity of 1 and -7 are odd.

The parity of integers is a simple concept. But based on parity, sometimes we can prove interesting or even surprising results.

Proofs based on parity argument.

Example:

We are given an 8×8 checkerboard (each square is painted white and black alternately). If we remove the top-left and the bottom-right white squares, can we cover the remaining 62 squares by using 1×2 dominoes?

No. Because:

- Each domino covers 1 white and 1 black square.
- So the portion of the checkerboard covered by dominoes must have the same number of white and black squares.
- If we remove two white squares, there are 32 black and 30 white squares left. So it is impossible to cover them all.

Example:

If we remove the top-left and the bottom-left squares, can we cover the remaining 62 squares by using 1×2 dominoes?

Proofs based on parity argument.

A number game:

Ask a friend do the following: (He does these steps while you are NOT watching.)

- Pick any odd integer n.
- Write the numbers $1, 2, 3, \ldots, 2n$ on a piece of paper.
- Pick any two numbers i and j on the paper.
- Erase i and j.
- Write $|i - j|$ on the paper.
- Repeat these steps, until only one number remains.

At this point, you will guess the parity of the remaining number.

- Hint: You always say "odd".
- Why you are always right?
- Try to make this game more interesting.

Proofs based on parity argument.

Fact:

The number of people (in this room) who has an odd number of friends (in this room) must be even.

We will proof this fact by using parity arguments.

Fact:

Among any 6 people, there must be:

- Either three mutual friends.
- Or three mutual strangers.

Try to prove this fact. (Not by parity argument).