Function, Sequence and Summation

CSE 191, Class Note 06
Computer Sci & Eng Dept
SUNY Buffalo

Function

Suppose A and B are nonempty sets. A function from A to B is an assignment of exactly one element of B to each element of A.

- We write $f : A \to B$.
- We write $f(a) = b$ if b is the element of B assigned to element a of A.

Example:

$f : Z \to Z$, where for each $x \in Z, f(x) = x^2$.

Domain, codomain, range

Suppose f is a function from A to B.

- We say A is the domain of f.
- We say B is the codomain of f.
- We say $\{f(x) \mid x \in A\}$ is the range of f.

Example:

$f : Z \to Z$, where for each $x \in Z$, $f(x) = x^2$.

- Domain of f : Z
- Codomain of f : Z
- Range of f : $\{x \mid x = y^2,\ y \in Z\}$

Image and preimage

Suppose f is a function from A to B and $f(x) = y$.

- We say y is the image of x.
- We say x is a preimage of y.

Note that the image of x is unique. But there can be more than one preimages for y.

Example:

$f : Z \to Z$, where for each $x \in Z$, $f(x) = x^2$.

- Image of 2: 4
- Preimage of 4: 2
- Another preimage of 4: -2

Image and preimage

- Note that every element in the domain has an image.
- But not every element in the codomain has a preimage.
- Only those in the range have preimages.

Example:

$f : Z \to Z$, where for each $x \in Z,\ f(x) = x^2$.

- Each $x \in Z$ has an image $f(x) = x^2$.
- But negative integers in Z do not have preimages.
- Only perfect squares (i.e., those in the range) have preimages.

One-to-one function

- A function is one-to-one if each element in the range has a unique preimage.
- Formally, $f : A \to B$ is one-to-one if $f(x) = f(y)$ implies $x = y$ for all $x \in A,\ y \in A$. Namely:

$$\forall x \in A\ \forall y \in A\ ((f(x) = f(y)) \to (x = y))$$

Example:

$f : Z \to Z$, where for each $x \in Z,\ f(x) = x^2$.

f is NOT one-to-one because 4 has two preimages.

Examples of one-to-one functions

- $f : N \to Z$, where for each $x \in N$, $f(x) = x + 5$.
- $f : Z^+ \to Z^+$, where for each $x \in Z^+$, $f(x) = x^2$.
- $f : \{0, 1, 2\} \to \{0, 1, 2, 3\}$, where $f(0) = 1$, $f(1) = 3$, $f(2) = 2$.

Onto function

- A function is onto if each element in codomain has a preimage (i.e., codomain = range).
- Formally, $f : A \to B$ is onto if for all $y \in B$, there is $x \in A$ such that $f(x) = y$. Namely:

$$\forall y \in B\ \exists x \in A\ (f(x) = y)$$

Example:

$f : Z \to Z$, where for each $x \in Z$, $f(x) = x^2$.

f is NOT onto because 2 does not have any preimage.

Examples of onto functions

- $f : R \to R^+ \cup \{0\}$, where for each $x \in R$, $f(x) = x^2$.
- $f : N \to Z^+$, where for each $x \in N$, $f(x) = x + 1$.
- $f : \{0, 1, 2, 3\} \to \{0, 1, 2\}$, where $f(0) = 1,\ f(1) = 1,\ f(2) = 2,\ f(3) = 0$.

Sum of functions

Suppose $f_1, f_2, \ldots, f_n$ are functions from A to R. The sum of $f_1, f_2, \ldots f_n$ is also a function from A to R defined as follows:

$$(f_1 + f_2 + \ldots + f_n)(x) = f_1(x) + f_2(x) + \ldots + f_n(x)$$

Example:

$f, g : R \to R$, where for each $x \in R$, $f(x) = x + 5;\ g(x) = x - 3$.

Then, $f + g$ is defined as $(f + g)(x) = (x + 5) + (x - 3) = 2x + 2$.

Product of functions

Suppose $f_1, f_2, \ldots, f_n$ are functions from A to R. The product of $f_1, f_2, \ldots, f_n$ is also a function from A to R defined as follows:

$$(f_1 f_2 \ldots f_n)(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x)$$

Example:

$f, g : R \to R$, where for each $x \in R$, $f(x) = x + 5; g(x) = x - 3$.

Then, fg is defined as $(f\ g)(x) = (x + 5)(x - 3) = x^2 + 2x - 15$.

Bijection

A function is a bijection if it is both one-to-one and onto. (It is also called a one-to-one correspondence).

Examples:

- Consider $f : R \to R$, where for each $x \in R$, $f(x) = 3x^2 - 5$.
  This is NOT a bijection because it is not one-to-one. For example, $f(1) = f(-1)$.
- Consider $f : R - \{1\} \to R$, where for each $x \in R$, $f(x) = x/(x - 1)$.
  This is NOT a bijection either, because it is not onto. For example, there is no x such that $f(x) = 1$.
- Consider $f : R \to R$, where for each $x \in R$, $f(x) = x^3 + 2$.
  This is a bijection.

Inverse function

Suppose f is a bijection from A to B. The inverse function of f is the function from B to A that assigns element b of B to element a of A if and only if $f(a) = b$.

- We use f^{-1} to represent the inverse of f.
- Hence, $f^{-1}(b) = a$ if and only if $f(a) = b$.

Example:

Consider $f : R^+ \to R^+$ where for each $x \in R^+$, $f(x) = x^2$.

Its inverse function is $g : R^+ \to R^+$, where for each $x \in R^+$, $g(x) = \sqrt{x}$.

Examples of inverse functions

Example

Consider $f : R^+ \to R^+$ where for each $x \in R^+$, $f(x) = 4x + 3$.

What is f^{-1}?

Example

Consider $f : \{0, 1, 2\} \to \{0, 1, 2\}$, where $f(0) = 1$, $f(1) = 2$, $f(2) = 0$.

What is f^{-1}?

Function composition

Suppose g is a function from A to B, and f is a function from B to C. Then the composition of f and g is a function from A to C defined as:

$$(f \circ g)(x) = f(g(x)).$$

Example:

Consider $f : R \to R$, where for each $x \in R$, $f(x) = 2x + 3$, and $g : R \to R$, where for each $x \in R$, $g(x) = 3x - 2$.

Then,

$$(f \circ g)(x) = f(3x - 2) = 2(3x - 2) + 3 = 6x - 1$$

Example composition

Example

$f : \{0, 1, 2\} \to \{0, 1, 2\}$, where $f(0) = 1,\ f(1) = 2,\ f(2) = 0$;
$g : \{0, 1, 2\} \to \{1, 2, 3\}$, where $g(0) = 1,\ g(1) = 2,\ g(2) = 3$.

What is $g \circ f$?

Incommutability of composition

Consider $f : R \to R$, where $f(x) = x + 1$, and $g : R \to R$, where $g(x) = x^2$. Then:

$$f \circ g(x) = x^2 + 1 \quad g \circ f(x) = (x + 1)^2$$

Clearly, $f \circ g \neq g \circ f$.

Caution:

In general, function composition is NOT commutable, which means the order of arguments in composition is important.

Graph of function

We can often draw a graph for a function $f : A \to B$: For each $x \in A$, we draw a point $(x, f(x))$ on the 2D plane. Typically, we need A and B to be subsets of R.

The graph of some important functions:

- Linear function $f(x) = kx + b$: a line
- Constant function $f(x) = c$: a line parallel to the X-axis
- Quadratic function $f(x) = ax^2 + bx + c$: parabola

Sequence

A sequence is a function whose domain is a set of integers.

- The domain is typically Z^+ (or, sometimes, N).
- The image of n is a_n.
- Each image a_n is called a term.
- For convenience, we often write it as $a_1, a_2, \ldots$ or $\{a_n\}$.

Example:

$1, 4, 9, 16, 25, \ldots$ is a sequence, where the nth term is $a_n = n^2$.

Example sequence

- $2, 9, 28, 65, \ldots$, where the nth term is $a_n = n^3 + 1$.
- $0, -2, -6, -12, \ldots$, where the nth term is $a_n = -n(n-1)$.
- $0, 1/2, 2/3, 3/4, \ldots$, where the nth term is $a_n = 1 - 1/n$.
- $-1, 1, -1, 1, \ldots$, where the nth term is $a_n = (-1)^n$.

Example questions for sequences

Example

What is the term a_4 of the sequence $\{a_n\}$ if $a_n = -(-2^n + n)$?

Solution: $a_4 = -(-2^4 + 4) = 12$.

Example

What is the term a_4 of the sequence $\{a_n = 5x + 3\}$?

Solution: $a_4 = 5x + 3$.
Note that each term is a function in x, and is independent from n.

Arithmetic sequence

- An arithmetic sequence is a sequence of the form $a, a + d, a + 2d, \ldots$.
- Formally, it is a sequence $\{a_n\}$, where $a_n = a + (n - 1)d$.
- Here a is called the initial term, d is called the common difference.

Example

$9, 4, -1, -6, \ldots$ is an arithmetic sequence, because it is of the form $a_n = 9 - 5(n - 1)$. The initial term is 9, and the common difference is -5.

Example of arithmetic sequence

Example:

Let x and y be two real numbers. Consider a sequence $\{a_n\}$, where $a_n = 5xn + 3y$.
Is this an arithmetic sequence?

The answer is yes, because we can rewrite it as $a_n = (5x + 3y) + 5x(n - 1)$. The initial term is $5x + 3y$. The common difference is $5x$.

Geometric sequence

- A geometric sequence is a sequence of the form $a, ar, ar^2, \ldots$.
- Formally, it is a sequence $\{a_0, a_1, \ldots, a_n, \ldots\}$, where $a_n = ar^n$.
- Here a is called the initial term, r is called the common ratio.

Example:

$9, 3, 1, 1/3, \ldots$ is an geometric sequence, because it is of the form $a_n = 9(1/3)^n$. The initial term is 9, and the common ratio is 1/3.

Example of geometric sequence

Example:

Let $x \neq 1$ be a real number. Consider a sequence $\{a_n\}$, where $a_n = x^{2n+5}$. Is this a geometric sequence?

- The answer is yes, because we can rewrite it as $a_n = x^5 x^{2n} = x^5 (x^2)^n$.
- The initial term is x^5.
- The common ratio is x^2.

Sum of terms

Given a sequence $\{a_n\}$, we can sum up its mth through nth terms. We write this sum as

$$\sum_{i=m}^{n} a_i$$

Note it is just a simplified way to write $a_m + a_{m+1} + \ldots + a_n$. There is no difference in meaning. Here i is called the index of the summation, m is called the lower limit of the index, and n is called the upper limit of the index.

Useful rules for $\sum$

- $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$.
- $\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$, where c is a constant.

Sum of arithmetic sequence

We often need to find the sum of the first n terms of a sequence. For example, consider an arithmetic sequence $\{a_n\}$, where $a_n = a + (n-1)d$. We have:

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \sum_{i=1}^{n}(a + d(i-1)) \\ &= \sum_{i=1}^{n} a + d\sum_{i=1}^{n}(i-1) \\ &= a \cdot n + d[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1] = a \cdot n + d\sum_{i=1}^{n} i - d \cdot n\end{aligned}$$

So the main problem is to find $\sum_{i=1}^{n} i$.

Sum of arithmetic sequence

What is $\sum_{i=1}^{n} i$?

- Let us denote it by $S = \sum_{i=1}^{n} i = 1 + 2 + \ldots + n$.
- We can change order of terms: $S = n + \ldots + 2 + 1$.
- Adding up the above two equations, we get: $2S = (n+1) + (n+1) + \ldots + (n+1) = n(n+1)$
- We get $S = n(n+1)/2$.

Sum of arithmetic sequence

Now we come back to the sum of the first n terms of arithmetic sequence:

Theorem:

$$\sum_{i=1}^{n} a_i = na + n(n-1)d/2$$

$$\begin{aligned}\sum_{i=1}^{n} a_i &= na + d(\sum_{i=1}^{n} i - n)\\ &= na + d(n(n+1)/2 - n)\\ &= na + n(n-1)d/2\end{aligned}$$

- This is an important formula for the sum of arithmetic sequence. You should memorize it.
- It is useful if you know the first term, the number of terms and the common difference, (but not the last term).

Sum of arithmetic sequence

Another important formula for the sum of arithmetic sequence:

Theorem:

$$\sum_{i=1}^{n} a_i = \frac{(a_1 + a_n) \cdot n}{2}$$

- We will prove this formula in class.
- It is useful, when you know the first and the last term, the number of terms, (but not the common difference).

Example of sum of arithmetic sequence

Consider the arithmetic sequence $9, 4, -1, -6, \ldots$. The sum of the first n terms is:

$$\sum_{i=1}^{n} a_i = 9n - 5n(n-1)/2$$

Thus the sum of the first 10 terms is:

$$9 \times 10 - 5 \times 10 \times 9/2 = 90 - 225 = -135$$

Note the 10th term is $9 - 5 \times 9 = -36$. By using the second formula, the sum of the first 10 terms is: $[9 + (-36)] \cdot 10/2 = -135$.

Sum of geometric sequence

Now consider a geometric sequence $\{a_0, a_1, \ldots, a_n, \ldots\}$,

$$\sum_{i=0}^{n} a_i = a \sum_{i=0}^{n} r^i$$

So the main problem is to find $\sum_{i=0}^{n} r^i$.

Sum of geometric sequence

What is $\sum_{i=0}^{n} r^i$?

- Let us denote it by $S = \sum_{i=0}^{n} r^i$.
- Recall its definition: $S = 1 + r + \ldots + r^n$
- We multiply both sides by r: $rS = r + r^2 \ldots + r^{n+1}$
- Taking the difference of two equations, we get:

$$(r-1)S = r^{n+1} - 1$$

Assuming $r \neq 1$, we get:

$$S = (r^{n+1} - 1)/(r-1)$$

Example of sum of geometric sequence

Now we come back to the sum of first n terms of geometric sequence:

$$\sum_{i=0}^{n} a_i = a \sum_{i=0}^{n} r^i = a \frac{r^{n+1} - 1}{r - 1}$$

- The above is the important formula for the sum of geometric sequence. You should memorize it.
- Keep in mind that this formula assumes $r \neq 1$.
- Question: what is the formula for $r = 1$?

Example sum of geometric sequence

Consider the geometric sequence 9, 3, 1, 1/3 .... The sum of first $n+1$ terms $a_0 + \cdots + a_n$ is:

$$\sum_{i=0}^{n} a_i = 9 \times \frac{(1/3)^{n+1} - 1}{1/3 - 1}$$
$$= \frac{27}{2} \times (1 - \frac{1}{3^{n+1}})$$

Thus the sum of the first 5 terms (i.e $n = 4$) is:

$$\frac{27}{2} \times (1 - \frac{1}{3^5}) = \frac{121}{9}$$

More example of sequence sum

Find the sum of: $\sum_{j=0}^{4}(3^j + 5 \cdot 2^j)$.
Solution:

$$\sum_{j=0}^{4}(3^j + 5 \cdot 2^j)$$
$$= \sum_{j=0}^{4} 3^j + \sum_{j=0}^{4} 5 \cdot 2^j$$
$$= \sum_{j=0}^{4} 3^j + 5 \cdot \sum_{j=0}^{4} 2^j$$
$$= \frac{3^5 - 1}{3 - 1} + 5 \cdot \frac{2^5 - 1}{2 - 1}$$
$$= 364 + 5 \cdot 63 = 679$$

Cardinality of Infinite Sets

Example 1

Consider two sets $A = \{a, b, c\}$ and $B = \{3, 9, 25\}$. Which set contains more elements?

Since each of A and B has 3 elements, they have equal size (cardinality).

Example 2

Consider two sets $A = \{a, b, c\}$ and $C = \{dog, cat, buffalo, wolf\}$. Which set contains more elements?

Since A has 3 elements and C has 4 elements, C has larger cardinality.

Example 3

Consider two sets $Z^+ = \{1, 2, 3, \ldots\}$ and $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Which set contains more elements?

Since both Z^+ and Z have infinitely many elements, we cannot really tell. Obviously, Z has larger cardinality than Z^+. Right? You will be surprised.

Cardinality of Infinite Sets

Definition

- The two sets A and B have the same cardinality if and only if there is a one-to-one and onto function (namely bijection) from A to B. In this case, we write $|A| = |B|$.
- If there is a one-to-one (not necessarily onto) function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- If $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$. This happens when
  - There exists an one-to-one function from A to B, and
  - There exists no one-to-one and onto function from A to B.

Cardinality of Infinite Sets

Definition

A set that is either finite, or has the same cardinality as the set of positive integers is called countable. When an infinite set S is countable, we write $|S| = \aleph_0$ (aleph null).

Intuitively, $|S| = \aleph_0$ if we can write: $S = \{a_1, a_2, a_3, \ldots a_i, \ldots\}$ such that:

- There's a rule that tells us which is a_i;
- Following this rule, we will reach every element in S.

Countable sets

Example

$|Z| = \aleph_0$. That is, the cardinality of Z^+ and Z are the same.

The following function f is a one-to-one correspondence (namely one-to-one and onto function) from Z^+ to Z:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

The function f is illustrated below:

Z^+	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Z	0	-1	1	-2	2	-3	3	-4	...

We need to show the function f is one-to-one and onto. This can be done by case-by-case proof.

Countable sets

Let $Q^+ = \{\ a/b \mid a$ and $b \neq 0$ are positive integers $\}$. Namely Q^+ is the set of positive rational numbers.

Example

$|Q^+| = \aleph_0$. That is, the cardinality of Q^+ and Z^+ are the same.

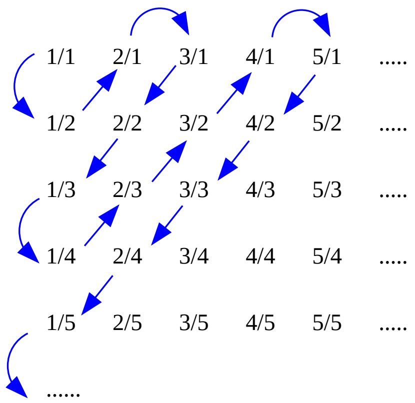

Figure: The 1-to-1 correspondence from Z^+ to Q^+.

Some Basic Lemmas about Cardinality

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Lemma

If $A \subseteq B$, then $|A| \leq |B|$.

Proof: Let $f : A \rightarrow B$ be the identity function from A to B. Namely $\forall x \in A,\ f(x) = x$. Then f is a 1-to-1 function, (not necessarily onto.) Thus by definition, $|A| \leq |B|$.

Schröder-Bernstein Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

This Theorem really says: If there is a 1-to-1 function $f : A \rightarrow B$ AND there is a 1-to-1 function $h : B \rightarrow A$, then there is a bijection from A to B. Its proof is NOT easy!

Some Basic Lemmas about Cardinality

Lemma

If both A and B are countable, then $A \cup B$ is also countable.

Lemma

Let $A_1, A_2, A_3 \cdots$ be a countable sequence, and each A_i is a countable set, then $A = \bigcup_{i=1}^{\infty} A_i$ is also countable.

Lemma

If A is countable, then for any integer $k \geq 1$, $A^k = A \times \cdots \times A$ (k times) is also countable.

We will prove these lemmas in class.

Definition

A set A is uncountable if $|A| > |Z^+|$.

So an uncountable set A contains really more elements than the set of positive integers.

Uncountable sets

Given the basic lemmas stated before, is there any set that is uncountable?

Theorem

Let A be any set. Let $\mathcal{P}(A)$ be the power set of A. Then $|A| < |\mathcal{P}(A)|$.

This Theorem says: there is no bijection from A to $\mathcal{P}(A)$. It is enough to show: for ANY function $f : A \to \mathcal{P}(A)$, f CANNOT be onto. We will prove this in class.

Example

Consider $A = \{a, b\}$. Then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Note that $|A| = 2 < 4 = |\mathcal{P}(A)|$.

Example

Consider Z^+. Then $\aleph_0 = |Z^+| < |\mathcal{P}(Z^+)|$.

The cardinality of $\mathcal{P}(Z^+)$ is denoted by $\aleph_1$.

Uncountable sets

- Similarly, we have $\aleph_1 = |\mathcal{P}(Z^+)| < |\mathcal{P}(\mathcal{P}(Z^+))| = \aleph_2$.
- We have an infinite sequence of increasing cardinalities:

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots \aleph_i < \ldots$$

here $\aleph_i$ is the cardinality of the i^{th} power set of Z^+.

You might feel these examples are too abstract. We have a simpler one.

Uncountable sets

Theorem

Let $(0, 1)$ be the set of real numbers between 0 and 1. Then $(0, 1)$ is uncountable.

Theorem

$|(0, 1)| = \aleph_1 = |\mathcal{P}(Z^+)|$

Theorem

Let R be the set of real numbers. Then $|(0, 1)| = |R|$.

The cardinality of R, which is also the cardinality of $(0, 1)$, is denoted by $\mathcal{C}$.
The proofs of these theorems will be given in class.

Uncountable sets

Now we have:

$$\aleph_0 < \aleph_1 = \mathcal{C} < \aleph_2 < \ldots \aleph_i < \ldots$$

- Is there a cardinality strictly between $\aleph_0$ and $\aleph_1$?

Uncountable sets

Hilbert's First Problem, also known as Continuum Hypothesis:
There is no cardinality $\mathcal{X}$ such that $\aleph_0 < \mathcal{X} < \aleph_1$.

- In 1900, David Hilbert (one of the greatest mathematician of his time) presented 23 unsolved problems to the mathematicians of the 20th century.
- All these problems are extremely hard.
- Some of Hilbert's problems have been solved. Some are not.
- Hilbert's first problem remains unsolved.
- Actually, there are indications that this is one proposition that can neither be proved to be true, nor to be false within our logic inference system for set theory.