Introduction to Relations

CSE 191, Class Note 09
Computer Sci & Eng Dept
SUNY Buffalo

Outline

Binary relation

Definition:

Let A and B be two sets. A binary relation from A to B is a subset of $A \times B$.

- In other words, a binary relation from A to B is a set of pair (a, b) such that $a \in A$ and $b \in B$.
- We often omit "binary" if there is no confusion.
- If R is a relation from A to B and $(a, b) \in R$, we say a is related to b by R. We write $a \; R \; b$.

Example:

Let A be the set of UB students, and B be the set of UB courses. Define relation R from A to B as follows: $a \; R \; b$ if and only if student a takes course b.

- So for every student x in this classroom, we have that $(x, CSE191) \in R$, or equivalently, $x \; R \; CSE191$.
- For your TA Ding , we know that Ding is NOT related to CSE191 by R.

Relation on a set

- We are particularly interested in binary relations from a set to the same set.
- If R is a relation from A to A, then we say R is a relation on set A.

Example:

We can define a relation R on the set of positive integers such that $a\ R\ b$ if and only if $a \mid b$.

$3\ R\ 6$. And 13 is not related to 6 by R.

Example:

We can define a relation R on the set of real numbers such that $a\ R\ b$ if and only if $a > b + 1$.

- Is 2 related to 3? Is 5 related to 3?
- For what values of x is x^2 related to $2x$?

Reflexive relation

Definition:

A relation R on a set A is called reflexive if every $a \in A$ is related to itself.

Example:

We can define a relation R on the set of positive integers such that $a \; R \; b$ if and only if $a \mid b$.

This relation is reflexive because every positive integer divides itself.

Example:

Consider the following relations on the set $\{1, 2, 3\}$:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3)\}$

$R_2 = \{(1, 1), (1, 3), (2, 2), (3, 1)\}$

$R_3 = \{(2, 3)\}$

$R_4 = \{(1, 1)\}$

Which of them are reflexive?

Symmetric and antisymmetric relation

Definition:

- A relation R on set A is called symmetric if $(a, b) \in R$ whenever $(b, a) \in R$.
- It is called antisymmetric if for all $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$.

Example:

We can define a relation R on the set of positive integers such that $a\ R\ b$ if and only if $a \mid b$.

- This relation is NOT symmetric because, e.g., $2\ R\ 4$ but 4 is not related to 2 by R.
- This relation is antisymmetric because $a|b$ and $b|a$ implies that $a = b$.

More examples

Example:

Consider the following relations on the set $\{1, 2, 3\}$:
$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,3)\}$
$R_2 = \{(1,1),(1,3),(2,2),(3,1)\}$
$R_3 = \{(2,3)\}$
$R_4 = \{(1,1)\}$

Which of them are symmetric?
Which of them are antisymmetric?

Transitive relation

Definition:

A relation R on set A is transitive if for all $a, b, c \in A$, $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.

Example:

We can define a relation R on the set of positive integers such that $a \; R \; b$ if and only if $a \mid b$.

This relation is transitive because $a \mid b$ and $b \mid c$ implies that $a \mid c$.

Example:

Consider the following relations on the set $\{1, 2, 3\}$:
$R_1 = \{(1, 1), (1, 2), (2, 3)\}$
$R_2 = \{(1, 2), (2, 3), (1, 3)\}$

Which of them is transitive?

Example relations and properties

Let R be the relation on the set of students in this class. Decide whether R is:

- reflexive (r/not r),
- symmetric (s/not s),
- antisymmetric (a/not a),
- transitive (t/not t).

1. $(a, b) \in R$ if and only if a has a higher GPA than b.

   not r; not s; a; t.

2. a and b have the same last name.

   r; s; not a; t.

3. a and b have a common hobby.

   r; s; not a; not t.

Outline

Composite relation

Definition:

Let R be a relation from set A to set B. Let S be a relation from set B to set C. The composite of R and S is a relation from A to C is:

$$\{(a,c) \mid \text{there exists } b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$$

We write it as $S \circ R$ (NOT $R \circ S$).

Example:

- We can define a relation R on the set of positive integers such that $a\ R\ b$ if and only if $a \mid b$.
- We also define relation S on the same set such that $b\ S\ c$ if and only if $b = 2c$.
- Let $T = S \circ R$. We have $a\ T\ c$ if and only if $a \mid 2c$.

Power relation

Definition

Let R be a relation on set A. For positive integer n, we define the nth power of R as follows:

- First, $R^1 = R$.
- Second, for all positive integer n, $R^{n+1} = R^n \circ R$.

This definition actually means:

- $R^2 = R \circ R$
- $R^3 = (R \circ R) \circ R$
- $R^4 = ((R \circ R) \circ R) \circ R$

$\ldots$

Example for power relation

Example:

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$ be a relation on $\{1,2,3,4\}$.

We can calculate the powers as follows:

- $R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$
- $R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

  $\ldots$

Keep in mind: $(a,c) \in R^n$ if and only if there exist $b_1, b_2, \ldots, b_{n-1}$, such that $(a,b_1),(b_1,b_2),\ldots,(b_{n-1},c) \in R$.

Transitive relation and power

Theorem:

A relation R on set A is transitive if and only if for all positive integer n, $R^n \subseteq R$.

Proof: First we show the "**if**" part.
Suppose for all positive integer n, $R^n \subseteq R$. In particular, $R^2 \subseteq R$. For all $a, b, c \in A$ such that $(a, b), (b, c) \in R$, we always have $(a, c) \in R^2$.
Consequently, $(a, c) \in R$. This means R is transitive.

Next, we prove the "**only if**" part.
Suppose R is transitive. We prove $R^n \subseteq R$ by induction.
When $n = 1$, clearly $R^1 = R \subseteq R$.

For the inductive step, we assume $R^n \subseteq R$ and try to establish that $R^{n+1} \subseteq R$.
For any $(a, c) \in R^{n+1}$, there exists $b \in A$ such that $(a, b) \in R^n$ and $(b, c) \in R$.
By the inductive assumption, we get that $(a, b) \in R$ and $(b, c) \in R$. Since R is transitive, we have that $(a, c) \in R$.
The above tells us that $R^{n+1} \in R$.

n-ary Relations and their Applications in Computer Science

Definition:

Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

Example:

Let R be the relation on $Z \times Z \times Z$ consisting of all triples of integers (a, b, c) such that a, b, c form an arithmetic progression.
Namely $(a, b, c) \in R$ if and only if $b - a = c - b$:

$$R = \{(a, b, c) \mid a \in Z,\ b \in Z,\ c \in Z,\ \text{and } b - a = c - b\}$$

- $(1, 3, 5) \in R$.
- $(-1, -11, -21) \in R$.
- $(1, 3, 7) \notin R$.

n-ary Relations and their Applications in Computer Science

- The concept of n-ary relations plays a central role in computer science. Especially in Database Systems.
- A database D consists of records, which are n-tuples, made up of fields.

Example:

A database D of student records may be made up of fields containing: the name, student ID, major, GPA.

Student Datadase

Name	ID	major	GPA
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Mathematics	3.49
...			

n-ary Relations and their Applications in Computer Science

- In this example, $R \subset$ Names $\times$ ID $\times$ Major $\times$ GPA. is a 4-ary relation. Each student record is a member of R.
- All database systems consist of such relations. (Examples: Sales record, inventory, factory order record, ...)
- One can define operators on n-ary relations, such as: selection operator, projection operator, join operators.
- These operators are used to construct new relations from existing relations in the database.
- They can also be used to retrieve information from the database records.
- Such a database system is called a relational database.
- SQL (short for Structured Query Language) is a database query language that uses these operators to perform database operations.
- It is widely used in everyday practical applications.

Outline

Matrix representation of relation

- A relation is defined to be a set of pairs, but it can also be represented in other ways, e.g., by a matrix.
- A matrix is simply a rectangular array of numbers.
- If a matrix has n rows and m columns, we call it an $n \times m$ matrix.
- In a matrix M, the number at the intersection of its i^{th} row and th column is written as M_{ij}.

Example:

Below we define a 2×3 matrix M:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

So we have $M_{13} = 0$ and $M_{22} = 2$.

Matrix Addition

Definition:

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. The sum of A and B, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words

$$A + B = [a_{ij} + b_{ij}]$$

Example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ 3 & -1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

Matrix Multiplication

Definition:

Let $A = [a_{ij}]$ be an $m \times k$ matrix and $B = [b_{ij}]$ be a $k \times n$ matrix. The product of A and B, denoted by AB, is the $m \times n$ matrix $C = [c_{ij}]$ such that

$$c_{ij} = \sum_{t=1}^{k} a_{it}b_{tj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Example:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\cdot 2 + 0\cdot 1 + 4\cdot 3 & 1\cdot 4 + 0\cdot 1 + 4\cdot 0 \\ 2\cdot 2 + 1\cdot 1 + 1\cdot 3 & 2\cdot 4 + 1\cdot 1 + 1\cdot 0 \\ 3\cdot 2 + 1\cdot 1 + 0\cdot 3 & 3\cdot 4 + 1\cdot 1 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \end{bmatrix}$$

Matrix representation of relation

definition:

To represent a relation R from set $A = \{a_1, a_2, \ldots, a_m\}$ to set $B = \{b_1, b_2, \ldots, b_n\}$, we can use an $m \times n$ matrix $M_P = [m_{ij}]$ where:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example:

Let $A = \{1, 3, 5\}$ and $B = \{1, 2\}$. Let R be a relation from A to B and $R = \{(1, 1), (3, 2), (5, 1)\}$. Then the matrix to represent R is:

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example matrix representation

Example:

Let $A = \{1, 2, 3, 4\}$ and R be a relation on A. Suppose $R = \{(1, 2), (3, 4)\}$.

What is the matrix representation of R?

Example:

Let $A = \{1, 2, 3\}$ and R be a relation on A. Suppose the following matrix represents R:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What is R?

Conversion to / from matrix representation

In general, we can easily convert a relation to/from its matrix representation:

- To convert it to its matrix representation, simply assign 1 to every M_{ij} such that $(a_i, b_j) \in R$ and assign 0 to all other entries.
- To convert it from its matrix representation, starting from an empty R, simply add (a_i, b_j) to R for every $M_{ij} = 1$.

Matrix of composite relations

Definition:

Let A, B, C be three sets with $|A| = m, |B| = p, |C| = n$. Let R be a relation from A to B represented by the $m \times p$ matrix $M_R = [a_{ij}]$. Let S be a relation from B to C represented by the $p \times n$ matrix $M_S = [b_{ij}]$. Let $S \circ R$ be the composite relation from A to C. Then $S \circ R$ is represented by the $m \times n$ matrix $M_{S \circ R} = M_R \odot M_S = [c_{ij}]$ where:

$$c_{ij} = \vee_{l=1}^{p} (a_{il} \wedge b_{lj}) = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots (a_{ip} \wedge b_{pj})$$

Note that the matrix operator $\odot$ is similar to the matrix multiplication:

- replace operator $+$ by $\vee$
- replace operator $\times$ by $\wedge$

Example

Example:

$A = B = C = \{a, b, c\}$.

- $R = \{(a,a), (a,c), (b,a), (b,b)\}$
- $S = \{(a,b), (b,c), (c,a), (c,c)\}$

We have: $S \circ R = \{(a,a), (a,b), (a,c), (b,b), (b,c)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Fact:

Let R be a relation on set A. Suppose R is represented by the matrix M_R. Then for any $k \geq 1$, the relation R^k is represented by the matrix:

$$M_{R^k} = M_R^k = \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{k \text{ times}}$$

Outline

Representing Relations by Using Digraphs

Definition

A Directed Graph (or digraph) consists a set V of vertices (or nodes) together with a set E of edges (or arcs). Each edge $e = (a, b)$ is an ordered pair of vertices. The vertex a is called the initial vertex of e. The vertex b is called the terminal vertex of e.

Definition:

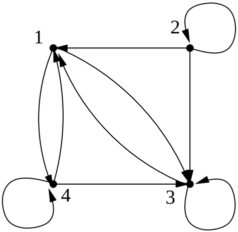
A relation R on a set A can be represented by a graph G_R:

- The vertex set of G_R is A.
- For each $(a, b) \in R$, there is a directed edge (a, b) in G_R.

Example

Example

The following graph G_R represents a relation on a set $A = \{1,2,3,4\}$ and $R = \{(1,3),(1,4)(2,1),(2,2),(2,3),(3,1),(3,3),(4,1),(4,3),(4,4)\}$.



Properties of Relations and their Graph Representation

Properties:

Let R be a relation on a set A. Let $G_R = (A, E)$ be the digraph representation of R.

- R is reflexive if and only if, for every vertex $a \in A$, there is a selfloop $(a, a) \in E$ in G_R.
- R is symmetric if and only if for every edge (a, b) in G_R, the edge (b, a) is also in G_R.
- R is antisymmetric if and only if for any two vertices $a \neq b$ in G, at most one of the two edges (a, b) and (b, a) is in G_R.
- R is transitive if and only if for any three vertices a, b, c G, $(a, b) \in E$ and $(b, c) \in E$ imply $(a, c) \in E$.

Outline

Closure

- Let A be a set and R be a relation on A. The reflexive / symmetric / transitive closure of R is a relation R^* such that:
  - R^* is a reflexive / symmetric / transitive relation on A.
  - $R \subseteq R^*$.
  - If R' is a reflexive / symmetric / transitive relation on A such that $R \subseteq R'$, then $R^* \subseteq R'$.
- Intuitively, a closure of R is the smallest extension of R to achieve a certain property (e.g., being reflexive / symmetric / transitive).

Example:

Suppose $A = \{1, 2, 3\}$. Suppose $R = \{(1, 2), (2, 3)\}$ is a relation on A.

- The reflexive closure of R is $\{(1, 2), (2, 3), (1, 1), (2, 2), (3, 3)\}$.
- The symmetric closure of R is $\{(1, 2), (2, 3), (2, 1), (3, 2)\}$.
- The transitive closure of R is $\{(1, 2), (2, 3), (1, 3)\}$.

Calculating closure

Given a set A and a relation R on A, how can we calculate the reflexive / symmetric / transitive closure of R?

- The reflexive closure of R can be calculated by adding (a, a) to R for every $a \in A$ (unless (a, a) is already in R).
- The symmetric closure of R can be calculated by adding (b, a) to R for every $(a, b) \in R$ (unless (b, a) is already in R).
- But calculating the transitive closure is more challenging. So we need to first have some theoretical analysis.

A technical lemma

Lemma 1:

Suppose A is a set $(|A| = n)$ and R is a relation on A. If for a positive integer k such that $(a, c) \in R^k$, then there exists a positive integer $m \leq n$ such that $(a, c) \in R^m$.

Proof: We prove this lemma by induction on k.
When $k \leq n$, the statement is trivially true. (Note this is a very strong initial step!)

Inductive step: Assume the statement is true for all positive integer $k \leq K$, where $K > n$. Now we show that it is also true when $k = K + 1$.

Suppose for a positive integer $K + 1$, $(a, c) \in R^{K+1}$.
We need to show there exists a positive integer $m \leq n$ such that $(a, c) \in R^m$.

A technical lemma

Since $(a,c) \in R^{K+1}$, there exist $b_1, \ldots, b_K$ such that $(a,b_1),(b_1,b_2),\ldots,(b_K,c) \in R$.
Consider the $K+1$ elements $b_1, b_2, \ldots, b_K$ and c. Since $K > n$, we have $K+1 > n+1$. Hence, among these elements, there must be two of them being equal to each other.

Without loss of generality, let us assume $b_i = b_j$ $(i < j)$. From $(a,b_1),\ldots,(b_{i-1},b_i) \in R$ and $(b_j,b_{j+1}),\ldots,(b_K,c) \in R$, we get that $(a,c) \in R^{K+1-(j-i)}$.

Since $j > i$, we know that $K+1-(j-i) < K+1$. Since $K+1-(j-i)$ is an integer, $K+1-(j-i) \leq K$. So, by the inductive assumption, there exists a positive integer $m < n$ such that $(a,c) \in R^m$.

A transitive relation R^*

Lemma 2:

Suppose A is a set $(|A| = n)$ and R is a relation on A. Then, $R^* = R \cup R^2 \cup \ldots \cup R^n$ is a transitive relation.

Proof: Consider any $a, b, c \in A$ such that $(a, b), (b, c) \in R^*$.
Then there exist i and j $(1 \leq i, j \leq n)$ such that $(a, b) \in R^i$, $(b, c) \in R^j$.
Then there exist $d_1, d_2, \ldots d_{i-1}, e_1, e_2, \ldots, e_{j-1}$ such that $(a, d_1), (d_1, d_2), \ldots, (d_{i-1}, b) \in R$, $(b, e_1), (e_1, e_2), \ldots, (e_{j-1}, c) \in R$. Hence, $(a, c) \in R^{i+j}$.
By Lemma 1, there exists positive integer $m \leq n$ such that $(a, c) \in R^m$. So, $(a, c) \in R^*$. This means R^* is transitive.

R^* contained in every transitive relation

Lemma 3:

Suppose A is a set $(|A| = n)$ and R is a relation on A. R' is a transitive relation on A such that $R \subseteq R'$. Then, $R^* = R \cup R^2 \cup \ldots \cup R^n \subseteq R'$.

Proof: For any $(a, c) \in R^*$, there exists positive integer $k \leq n$ such that $(a, c) \in R^k$.
Hence, there exist $b_1, b_2, \ldots, b_{k-1}$ such that $(a, b_1), (b_1, b_2), \ldots, (b_{k-1}, c) \in R$.
Since $R \subseteq R'$, we get that $(a, b_1), (b_1, b_2), \ldots, (b_{k-1}, c) \in R'$. Since R' is transitive, we get that $(a, c) \in R'$. This means $R^* \subseteq R'$.

Theorem:

Suppose A is a set $(|A| = n)$ and R is a relation on A. Then, $R^* = R \cup R^2 \cup \ldots \cup R^n$ is the transitive closure of R.

Proof: Clearly $R \subseteq R^*$. Lemma 2 tells us that R^* is transitive. Lemma 3 tells us that for any transitive relation R' such that $R \subseteq R'$, $R^* \subseteq R'$. Putting all these together, we see that R^* is the transitive closure of R.

Procedure for Computing the Transitive Closure

Let R be a relation. Let M_R be the matrix representation of R. The following procedure calculate the matrix representation B of the transitive closure R^* of R.

Procedure Transitive Closure(M_R)

1. $A := M_R$
2. $B := A$
3. for $i := 2$ to n
4. $\quad\quad A := A \odot M_R$
5. $\quad\quad B := B \vee A$
6. return B

Outline

Equivalence relation

Definition:

A relation R on set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

- Two elements related by an equivalent relation are called equivalent.
- When R is an equivalence relation, we often write $a \sim b$ instead of $a\ R\ b$.

Example:

Let R be a relation on the set of real numbers such that $a\ R\ b$ if and only if $|a| = |b|$. Then R is an equivalence relation.

Example for equivalence relation

Suppose the following relations are defined on the set of students in this class. Are they equivalence relations?

- $\{(a, b) \mid a$ and b were admitted to UB in the same year$\}$.
  Yes.
- $\{(a, b) \mid a$ and b have a common friend $\}$.
  No. (It is not transitive.)
- $\{(a, b) \mid a$ and b have played volleyball together $\}$.
  No. (It is not transitive.)

Example for equivalence relation

Which of the following relations are equivalence relation?

- Let R be a relation on the set of real numbers such that $a\ R\ b$ if and only if $a - b$ is an integer.
- Let R be a relation on the set of positive integers such that $a\ R\ b$ if and only if a divides b.
- Let c be a positive real number. Let R be a relation on the set of real numbers such that $a\ R\ b$ if and only if $|a - b| < c$.
- Let R be a relation on the set of UB students such that $a\ R\ b$ if and only if a and b are classmates in one of the courses taken in Fall 2008.
- Let R be a relation on the set of CSE191 students such that $a\ R\ b$ if and only if a and b are in the same recitation section.

Equivalence class

Definition:

Let R be an equivalence relation on set A. For $a \in A$, the set

$$\{x \mid x \sim a\}$$

is called the equivalence class of a.

- We often write it as $[a]_R$.
- When only one relation is under consideration, we can simply write it as $[a]$.

Example:

Let R be the equivalence relation on the set of real numbers such that $a\ R\ b$ if and only if $|a| = |b|$. Then, $[a]_R = \{a, -a\}$.

Example for equivalence class

- Let R be the equivalence relation on the set of real numbers such that $a \; R \; b$ if and only if $a - b$ is an integer.

  For real number a, what is its equivalence class?

- Let R be the equivalence relation on the set of CSE191 students such that $a \; R \; b$ if and only if a and b are in the same recitation section.

  For student a taking CSE191, what is $[a]_R$?

Properties of equivalence class

Theorem:

Let R be an equivalence class on set A. These statements for elements a, b of A are equivalent:

1. $a\ R\ b$;
2. $[a] = [b]$;
3. $[a] \cap [b] \neq \emptyset$.

Proof: We first show (1) implies (2).
We show $[a] \subseteq [b]$.
Consider any $x \in [a]$. We have $x\ R\ a$. Since R is a an equivalence relation, it is transitive. So since $a\ R\ b$, we have that $x\ R\ b$. Hence, $x \in [b]$. Therefore, $[a] \subseteq [b]$.
Symmetrically, we can show $[b] \subseteq [a]$.
Combining the above, we have that $[a] = [b]$.

Second, we see that (2) trivially implies (3).

Properties of equivalence class

Third, we prove (3) implies (1).
Since $[a] \cap [b] \neq \emptyset$, we can choose an element $x \in [a] \cap [b]$.
Clearly, $x \in [a]$ and $x \in [b]$. From $x \in [a]$ we get that $x\ R\ a$, and from $x \in [b]$ we get that $x\ R\ b$.
Since R is an equivalence relation, it is both symmetric and transitive. Since R is symmetric, we get that $a\ R\ x$. Since R is transitive, we get that $a\ R\ b$.

In summary, we have shown that

- (1) implies (2),
- (2) implies (3),
- (3) implies (1).

So (1) (2) (3) are all equivalent.

Outline

Partition

- The above theorem implies that, for any two elements a and b of A, we have either $[a] = [b]$, or $[a]$ and $[b]$ are disjoint.
- Intuitively, this means that the elements of A are partitioned into a number of disjoint sets.

Definition

Let A be a set. a partition P of A is a subset of $P(A)$ such that the following two conditions are met:

1. For any $S, T \in P$ $(S \neq T), S \cap T = \emptyset$.
2. $\bigcup_{S \in P} S = A$.

In other words, a partition of A is a set of subsets of A, such that these subsets are pairwise disjoint, and that the union of these subsets is equal to A.

Example for partition

Suppose $A = \{1, 2, 3, 4, 5\}$. Then:

- The set $\{\{2\}, \{1, 3\}, \{4, 5\}\}$ is a partition of A.
- The set $\{\{2\}, \{1, 2, 3\}, \{4, 5\}\}$ is not a partition of A, because $\{2\}$ and $\{1, 2, 3\}$ are not disjoint.
- The set $\{\{2\}, \{1, 3\}, \{5\}\}$ is not a partition of A, because the union of $\{2\}$, $\{1, 3\}$ and $\{5\}$ is $\{1, 2, 3, 5\}$, not equal to A.

Which of the following is a partition of the set of real numbers?

- The set $\{\{x \mid x - a \text{ is an integer}\} \mid a \text{ is a real number}\}$.
- The set $\{\{x \mid x - a > 0\} \mid a \text{ is a real number}\}$.
- The set $\{\{x \mid x - a \text{ is an integer}\} \mid a \text{ is an integer}\}$.

Equivalence relation and partition

Theorem:

(1) Let R be an equivalence relation on set A. Then the set of equivalent classes of R is a partition on A.
(2) Conversely, given a partition of A, there exists an equivalence relation on A such that this partition is the set of its equivalence classes.

Proof: (1) We have already shown that any two equivalence classes not equal to each other are disjoint. So all we need to show is that the union of equivalence classes is equal to A.
Clearly the union of equivalence classes is a subset of A.
For any $a \in A$, since R is an equivalence relation and hence reflexive, clearly we have that $a \ R \ a$, which means $a \in [a]$. So a is an element of the union of equivalence classes. Therefore, A is a subset of the union of equivalence classes.
Putting the above together, we see that the union of equivalence classes is equal to A.

Equivalence relation and partition

(2) Given a partition P of A, we define a relation R as follows:
For elements a and b of A, $a \; R \; b$ if and only if there exists $S \in P$ such that $a, b \in S$.

We first show that R is reflexive. Clearly, for any $a \in A$, since A is equal to the union of all elements of P, there exists an element S of P such that $a \in S$. Hence, $a \; R \; a$. Therefore, R is reflexive.

Second, we show that R is symmetric. For $a, b \in A$ such that i$a \; R \; b$, we have that there exists an element S of P such that $a, b \in S$. This is equivalent to say $b, a \in S$. So we have that $b \; R \; a$. Therefore, R is symmetric.

Third, we show that R is transitive.
For $a, b, c \in A$ such that $a \; R \; b$ and $b \; R \; c$, we have that there exist $S, T \in P$ such that $a, b \in S$ and $b, c \in T$.
Clearly, S and T are not disjoint. So, $S = T$.
Hence, $a, c \in T$, which means $a \; R \; c$. Therefore, R is transitive.

Combining the above three points, we get that R is an equivalence relation.

Example for equivalence relation and partition

Example:

Let R be the equivalence relation on the set of real numbers such that $a\ R\ b$ if and only if $|a| = |b|$.

Recall $[a]_R = \{a, -a\}$.
The set of equivalence classes $\{\{a, -a\} \mid a$ is a real number$\}$ is a partition of the set of real numbers.

Example:

Suppose $A = \{1, 2, 3, 4, 5\}$. Recall the set $\{\{2\}, \{1, 3\}, \{4, 5\}\}$ is a partition of A. Correspondingly, we have an equivalence relation

$$R = \{(2, 2), (1, 1), (3, 3), (1, 3), (3, 1), (4, 4), (5, 5), (4, 5), (5, 4)\}$$

It is easy to verify that the set of equivalence classes of R is $\{\{2\}, \{1, 3\}, \{4, 5\}\}$.

Outline

Partial order

Definition:

A relation R on set A is a partial order if it is reflexive, antisymmetric, and transitive.

- If R is a partial order on A, then (A, R) is called a poset. Sometimes, we just say A is a poset, if no ambiguity.
- Suppose R is partial order on A. If $a\ R\ b$ or $b\ R\ a$, then a and b is comparable; otherwise, a and b are incomparable.

Example:

$R = \{(a, b) \mid a \leq b,\ a, b$ are real numbers$\}$ is a partial order on the set of real numbers.

Example for partial order

Suppose the relations below are defined on the set $\{a, b, c, d\}$. Which is a partial order?

- $\{(a, a), (c, c), (d, d)\}$.
  No. (It is not reflexive.)
- $\{(a, a), (b, b), (c, a), (c, c), (c, d), (d, d)\}$
  Yes, it is a partial order.
- $\{(a, a), (b, b), (b, c), (c, c), (d, b), (d, d)\}$
  No. It's not transitive.

Example for partial order

Is relation R on the set of positive integers defined below a partial order?

- $R = \{(a,b) \mid a|b,\ a$ and b are positive integers $\}$.
- $R = \{(a,b) \mid b/a$ is an integer > 1, a and b are positive integers$\}$.
- $R = \{(a,b) \mid a|b$ or $b|a$, a and b are positive integers $\}$.
- $R = \{(a,b) \mid b/a = 2$, a and b are positive integers$\}$.

Consider partial order $R = \{(a,b) \mid a|b,\ a$ and b are positive integers$\}$ on the set of positive integers.

- Are 3 and 6 comparable?
- Are 3 and 5 comparable?
- Are 6 and 3 comparable?
- Are 6 and 4 comparable?

Total order

Definition:

If R is a partial order on set A and every two elements of A are comparable, then R is a total order on A.

Example:

$R = \{(a,b) \mid a \leq b,\ a,b$ are real numbers$\}$ is a total order on the set of real numbers, since for every two real numbers a and b, we must have either $a \leq b$ or $b \leq a$.

Example:

Recall relation $R = \{(a,b) \mid a|b,\ a$ and b are positive integers$\}$ is a partial order on the set of positive integer.
It is not a total order since 3 and 5 are not comparable.