TABLE 9 Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Information is often represented using bit strings, which are lists of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 12 101010011 is a bit string of length nine. ◄

We can extend bit operations to bit strings. We define the **bitwise *OR*, bitwise *AND***, and **bitwise *XOR*** of two strings of the same length to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$ to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 13.

EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

01 1011 0110
11 0001 1101
11 1011 1111 bitwise *OR*
01 0001 0100 bitwise *AND*
10 1010 1011 bitwise *XOR* ◄

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?
- **a)** Boston is the capital of Massachusetts.
- **b)** Miami is the capital of Florida.
- **c)** $2 + 3 = 5$.
- **d)** $5 + 7 = 10$.
- **e)** $x + 2 = 11$.
- **f)** Answer this question.

2. Which of these are propositions? What are the truth values of those that are propositions?
- **a)** Do not pass go.
- **b)** What time is it?
- **c)** There are no black flies in Maine.
- **d)** $4 + x = 5$.
- **e)** The moon is made of green cheese.
- **f)** $2^n \geq 100$.

3. What is the negation of each of these propositions?
- **a)** Mei has an MP3 player.
- **b)** There is no pollution in New Jersey.
- **c)** $2 + 1 = 3$.
- **d)** The summer in Maine is hot and sunny.

4. What is the negation of each of these propositions?
- **a)** Jennifer and Teja are friends.
- **b)** There are 13 items in a baker's dozen.
- **c)** Abby sent more than 100 text messages every day.
- **d)** 121 is a perfect square.

5. What is the negation of each of these propositions?
- **a)** Steve has more than 100 GB free disk space on his laptop.
- **b)** Zach blocks e-mails and texts from Jennifer.
- **c)** $7 \cdot 11 \cdot 13 = 999$.
- **d)** Diane rode her bicycle 100 miles on Sunday.

6. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.
- **a)** Smartphone B has the most RAM of these three smartphones.
- **b)** Smartphone C has more ROM or a higher resolution camera than Smartphone B.
- **c)** Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.
- **d)** If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.
- **e)** Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

7. Suppose that during the most recent fiscal year, the annual revenue of Acme Computer was 138 billion dollars and its net profit was 8 billion dollars, the annual revenue of Nadir Software was 87 billion dollars and its net profit was 5 billion dollars, and the annual revenue of Quixote Media was 111 billion dollars and its net profit was 13 billion dollars. Determine the truth value of each of these propositions for the most recent fiscal year.
- **a)** Quixote Media had the largest annual revenue.
- **b)** Nadir Software had the lowest net profit and Acme Computer had the largest annual revenue.
- **c)** Acme Computer had the largest net profit or Quixote Media had the largest net profit.
- **d)** If Quixote Media had the smallest net profit, then Acme Computer had the largest annual revenue.
- **e)** Nadir Software had the smallest net profit if and only if Acme Computer had the largest annual revenue.

8. Let p and q be the propositions

p : I bought a lottery ticket this week.
q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.
- **a)** $\neg p$ **b)** $p \vee q$ **c)** $p \to q$
- **d)** $p \wedge q$ **e)** $p \leftrightarrow q$ **f)** $\neg p \to \neg q$
- **g)** $\neg p \wedge \neg q$ **h)** $\neg p \vee (p \wedge q)$

9. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.
- **a)** $\neg q$ **b)** $p \wedge q$ **c)** $\neg p \vee q$
- **d)** $p \to \neg q$ **e)** $\neg q \to p$ **f)** $\neg p \to \neg q$
- **g)** $p \leftrightarrow \neg q$ **h)** $\neg p \wedge (p \vee \neg q)$

10. Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.
- **a)** $\neg p$ **b)** $p \vee q$
- **c)** $\neg p \wedge q$ **d)** $q \to p$
- **e)** $\neg q \to \neg p$ **f)** $\neg p \to \neg q$
- **g)** $p \leftrightarrow q$ **h)** $\neg q \vee (\neg p \wedge q)$

11. Let p and q be the propositions

p : It is below freezing.
q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).
- **a)** It is below freezing and snowing.
- **b)** It is below freezing but not snowing.
- **c)** It is not below freezing and it is not snowing.
- **d)** It is either snowing or below freezing (or both).
- **e)** If it is below freezing, it is also snowing.
- **f)** Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
- **g)** That it is below freezing is necessary and sufficient for it to be snowing.

12. Let p, q, and r be the propositions

p : You have the flu.
q : You miss the final examination.
r : You pass the course.

Express each of these propositions as an English sentence.
- **a)** $p \to q$ **b)** $\neg q \leftrightarrow r$
- **c)** $q \to \neg r$ **d)** $p \vee q \vee r$
- **e)** $(p \to \neg r) \vee (q \to \neg r)$
- **f)** $(p \wedge q) \vee (\neg q \wedge r)$

13. Let p and q be the propositions

p : You drive over 65 miles per hour.
q : You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).
- **a)** You do not drive over 65 miles per hour.
- **b)** You drive over 65 miles per hour, but you do not get a speeding ticket.
- **c)** You will get a speeding ticket if you drive over 65 miles per hour.
- **d)** If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
- **e)** Driving over 65 miles per hour is sufficient for getting a speeding ticket.
- **f)** You get a speeding ticket, but you do not drive over 65 miles per hour.
- **g)** Whenever you get a speeding ticket, you are driving over 65 miles per hour.

14. Let p, q, and r be the propositions

p : You get an A on the final exam.
q : You do every exercise in this book.
r : You get an A in this class.

Write these propositions using p, q, and r and logical connectives (including negations).

5. What is the negation of each of these propositions?
   a) Steve has more than 100 GB free disk space on his laptop.
   b) Zach blocks e-mails and texts from Jennifer.
   c) $7 \cdot 11 \cdot 13 = 999$.
   d) Diane rode her bicycle 100 miles on Sunday.

6. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.
   a) Smartphone B has the most RAM of these three smartphones.
   b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.
   c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.
   d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.
   e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

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   c) Acme Computer had the largest net profit or Quixote Media had the largest net profit.
   d) If Quixote Media had the smallest net profit, then Acme Computer had the largest annual revenue.
   e) Nadir Software had the smallest net profit if and only if Acme Computer had the largest annual revenue.

8. Let p and q be the propositions
   p : I bought a lottery ticket this week.
   q : I won the million dollar jackpot.

   Express each of these propositions as an English sentence.
   a) $\neg p$ b) $p \vee q$ c) $p \rightarrow q$
   d) $p \wedge q$ e) $p \leftrightarrow q$ f) $\neg p \rightarrow \neg q$
   g) $\neg p \wedge \neg q$ h) $\neg p \vee (p \wedge q)$

9. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.
   a) $\neg q$ b) $p \wedge q$ c) $\neg p \vee q$
   d) $p \rightarrow \neg q$ e) $\neg q \rightarrow p$ f) $\neg p \rightarrow \neg q$
   g) $p \leftrightarrow \neg q$ h) $\neg p \wedge (p \vee \neg q)$

10. Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.
   a) $\neg p$ b) $p \vee q$
   c) $\neg p \wedge q$ d) $q \rightarrow p$
   e) $\neg q \rightarrow \neg p$ f) $\neg p \rightarrow \neg q$
   g) $p \leftrightarrow q$ h) $\neg q \vee (\neg p \wedge q)$

11. Let p and q be the propositions
   p : It is below freezing.
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   Write these propositions using p and q and logical connectives (including negations).
   a) It is below freezing and snowing.
   b) It is below freezing but not snowing.
   c) It is not below freezing and it is not snowing.
   d) It is either snowing or below freezing (or both).
   e) If it is below freezing, it is also snowing.
   f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
   g) That it is below freezing is necessary and sufficient for it to be snowing.

12. Let p, q, and r be the propositions
   p : You have the flu.
   q : You miss the final examination.
   r : You pass the course.

   Express each of these propositions as an English sentence.
   a) $p \rightarrow q$ b) $\neg q \leftrightarrow r$
   c) $q \rightarrow \neg r$ d) $p \vee q \vee r$
   e) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$
   f) $(p \wedge q) \vee (\neg q \wedge r)$

13. Let p and q be the propositions
   p : You drive over 65 miles per hour.
   q : You get a speeding ticket.

   Write these propositions using p and q and logical connectives (including negations).
   a) You do not drive over 65 miles per hour.
   b) You drive over 65 miles per hour, but you do not get a speeding ticket.
   c) You will get a speeding ticket if you drive over 65 miles per hour.
   d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
   e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
   f) You get a speeding ticket, but you do not drive over 65 miles per hour.
   g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

14. Let p, q, and r be the propositions
   p : You get an A on the final exam.
   q : You do every exercise in this book.
   r : You get an A in this class.

   Write these propositions using p, q, and r and logical connectives (including negations).

24. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) I will remember to send you the address only if you send me an e-mail message.
b) To be a citizen of this country, it is sufficient that you were born in the United States.
c) If you keep your textbook, it will be a useful reference in your future courses.
d) The Red Wings will win the Stanley Cup if their goalie plays well.
e) That you get the job implies that you had the best credentials.
f) The beach erodes whenever there is a storm.
g) It is necessary to have a valid password to log on to the server.
h) You will reach the summit unless you begin your climb too late.

25. Write each of these propositions in the form "p if and only if q" in English.

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.

26. Write each of these propositions in the form "p if and only if q" in English.

a) For you to get an A in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.
b) If you read the newspaper every day, you will be informed, and conversely.
c) It rains if it is a weekend day, and it is a weekend day if it rains.
d) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him.

27. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows today, I will ski tomorrow.
b) I come to class whenever there is going to be a quiz.
c) A positive integer is a prime only if it has no divisors other than 1 and itself.

28. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows tonight, then I will stay at home.
b) I go to the beach whenever it is a sunny summer day.
c) When I stay up late, it is necessary that I sleep until noon.

29. How many rows appear in a truth table for each of these compound propositions?

a) $p \to \neg p$
b) $(p \vee \neg r) \wedge (q \vee \neg s)$
c) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$
d) $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$

30. How many rows appear in a truth table for each of these compound propositions?

a) $(q \to \neg p) \vee (\neg p \to \neg q)$
b) $(p \vee \neg t) \wedge (p \vee \neg s)$
c) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$
d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

31. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$
b) $p \vee \neg p$
c) $(p \vee \neg q) \to q$
d) $(p \vee q) \to (p \wedge q)$
e) $(p \to q) \leftrightarrow (\neg q \to \neg p)$
f) $(p \to q) \to (q \to p)$

32. Construct a truth table for each of these compound propositions.

a) $p \to \neg p$
b) $p \leftrightarrow \neg p$
c) $p \oplus (p \vee q)$
d) $(p \wedge q) \to (p \vee q)$
e) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

33. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \to (p \oplus q)$
b) $(p \oplus q) \to (p \wedge q)$
c) $(p \vee q) \oplus (p \wedge q)$
d) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
f) $(p \oplus q) \to (p \oplus \neg q)$

34. Construct a truth table for each of these compound propositions.

a) $p \oplus p$
b) $p \oplus \neg p$
c) $p \oplus \neg q$
d) $\neg p \oplus \neg q$
e) $(p \oplus q) \vee (p \oplus \neg q)$
f) $(p \oplus q) \wedge (p \oplus \neg q)$

35. Construct a truth table for each of these compound propositions.

a) $p \to \neg q$
b) $\neg p \leftrightarrow q$
c) $(p \to q) \vee (\neg p \to q)$
d) $(p \to q) \wedge (\neg p \to q)$
e) $(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$
f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$

36. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \vee r$
b) $(p \vee q) \wedge r$
c) $(p \wedge q) \vee r$
d) $(p \wedge q) \wedge r$
e) $(p \vee q) \wedge \neg r$
f) $(p \wedge q) \vee \neg r$

37. Construct a truth table for each of these compound propositions.

a) $p \to (\neg q \vee r)$
b) $\neg p \to (q \to r)$
c) $(p \to q) \vee (\neg p \to r)$
d) $(p \to q) \wedge (\neg p \to r)$
e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$
f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

38. Construct a truth table for $((p \to q) \to r) \to s$.

39. Construct a truth table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$.

saw both Smith and Jones with Cooper the day of the killing and that either Smith or Jones must have killed him. Can you determine who the murderer was if

a) one of the three men is guilty, the two innocent men are telling the truth, but the statements of the guilty man may or may not be true?

b) innocent men do not lie?

33. Steve would like to determine the relative salaries of three coworkers using two facts. First, he knows that if Fred is not the highest paid of the three, then Janice is. Second, he knows that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie, and Janice from what Steve knows? If so, who is paid the most and who the least? Explain your reasoning.

34. Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is known? Either Kevin or Heather, or both, are chatting. Either Randy or Vijay, but not both, are chatting. If Abby is chatting, so is Randy. Vijay and Kevin are either both chatting or neither is. If Heather is chatting, then so are Abby and Kevin. Explain your reasoning.

35. A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

36. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said "Carlos did it." John said "I did not do it." Carlos said "Diana did it." Diana said "Carlos lied when he said that I did it."

a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

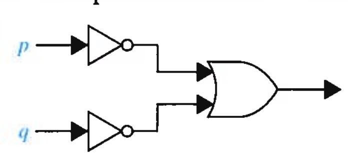
b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

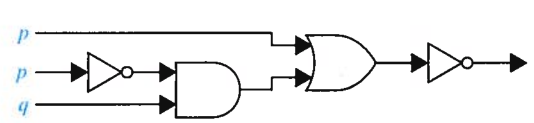
37. Suppose there are signs on the doors to two rooms. The sign on the first door reads "In this room there is a lady, and in the other one there is a tiger"; and the sign on the second door reads "In one of these rooms, there is a lady, and in one of them there is a tiger." Suppose that you know that one of these signs is true and the other is false. Behind which door is the lady?

***38.** Solve this famous logic puzzle, attributed to Albert Einstein, and known as the **zebra puzzle**. Five men with different nationalities and with different jobs live in consecutive houses on a street. These houses are painted different colors. The men have different pets and have different favorite drinks. Determine who owns a zebra and whose favorite drink is mineral water (which is one of the favorite drinks) given these clues: The Englishman lives in the red house. The Spaniard owns a dog. The Japanese man is a painter. The Italian drinks tea. The Norwegian lives in the first house on the left. The green house is immediately to the right of the white one. The photographer breeds snails. The diplomat lives in the yellow house. Milk is drunk in the middle house. The owner of the green house drinks coffee. The Norwegian's house is next to the blue one. The violinist drinks orange juice. The fox is in a house next to that of the physician. The horse is in a house next to that of the diplomat. [*Hint:* Make a table where the rows represent the men and columns represent the color of their houses, their jobs, their pets, and their favorite drinks and use logical reasoning to determine the correct entries in the table.]

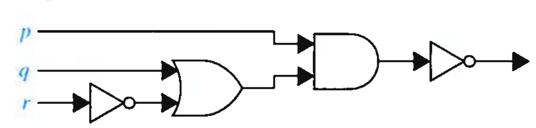
39. Freedonia has fifty senators. Each senator is either honest or corrupt. Suppose you know that at least one of the Freedonian senators is honest and that, given any two Freedonian senators, at least one is corrupt. Based on these facts, can you determine how many Freedonian senators are honest and how many are corrupt? If so, what is the answer?

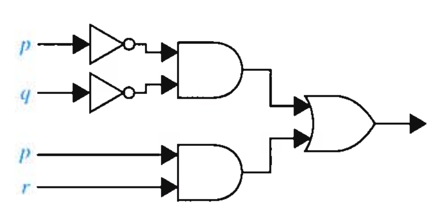
40. Find the output of each of these combinatorial circuits.

a) 

b) 

41. Find the output of each of these combinatorial circuits.

a) 

b) 

42. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $(p \wedge \neg r) \vee (\neg q \wedge r)$ from input bits p, q, and r.

43. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $((\neg p \vee \neg r) \wedge \neg q) \vee (\neg p \wedge (q \vee r))$ from input bits p, q, and r.

1.3 Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

We begin our discussion with a classification of compound propositions according to their possible truth values.

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Tautologies and contradictions are often important in mathematical reasoning. Example 1 illustrates these types of compound propositions.

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction. ◀

Logical Equivalences

Demo

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2 The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions p and q are equivalent if and only if the columns

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F