Solution #4, CSE 191

Fall, 2014

1. (0 points). Page 108, Prob 1. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

Sol: The proof is case-by-case.

$$n = 1, 1^2 + 1 \geq 2^1$$
$$n = 2, 2^2 + 1 \geq 2^2$$
$$n = 3, 3^2 + 1 \geq 2^3$$
$$n = 4, 4^2 + 1 \geq 2^4$$

2. (0 points). Page 108, Prob 6.

Sol: Since x and y are integers of opposite parity; it means either "x is odd and y is even" or "x is even and y is odd". Without loss of generality, we assume x is odd and y is even. So, $x + y$ is odd. In this problem, we want to prove $5x + 5y$ is odd. Since 5 and $x + y$ are odd numbers, $5x + 5y = 5(x + y)$ is also an odd number.

3. (0 points). Page 108, Prob 14.

Sol: Disprove that if a and b are rational number, then a^b is also rational.

Let $a = 2$ and $b = \frac{1}{2}$. Then $a^b = 2^{\frac{1}{2}} = \sqrt{2}$ is irrational.

4. (0 points). Page 108, Prob 24. Prove: $\forall x, y \in \mathrm{R}$, $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$.

Sol: $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \Leftrightarrow \frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2$ (because $\sqrt{\frac{x^2+y^2}{2}} \geq 0$).

This is equivalent to $\frac{x^2+y^2}{2} - (\frac{x+y}{2})^2 \geq 0$. But:

$$\frac{x^2 + y^2}{2} - \left(\frac{x + y}{2}\right)^2 = \frac{(x - y)^2}{4} \geq 0, \tag{1}$$

This proves the original innequality.

5. (0 points). Page 108, Prob 34.

Sol: Prove that $\sqrt[3]{2}$ is irrational. The proof is by contradiction. Suppose $\sqrt[3]{2} = \frac{b}{a}$ is rational where $a, b \in N$. Without loss of generality, we assume a and b have no common divisors. (If not, we can divide both a and b by their greatest common divisor).

$$\sqrt[3]{2} = \frac{b}{a}$$
$$\Leftrightarrow (\sqrt[3]{2})^3 = (\frac{b}{a})^3$$
$$\Leftrightarrow 2 = \frac{b^3}{a^3}$$
$$\Leftrightarrow 2a^3 = b^3$$

Since $2a^3$ is even, b^3 is also even. Thus $b = 2k$ for some integer k. Then $2a^3 = (2k)^3 = 8k^3$. This implies that $a^3 = 4k^3$. Since $4k^3$ is even, a must be even. It contradicts our assumption that a and b have no common divisors. So, $\sqrt[3]{2}$ must be irrational.

6. (0 points). Page 109, Prob 36.

Prof: Here we use I to denote the set of irrational numbers, and Q to denote the set of rational numbers. Now, we arbitrarily select $x \in Q$ and $y \in I$. WLOG, we assume $x < y$.

Let $z = (x + y)/2$. We show z is the number we are looking for.

First, note that z is the average of x and y. So we have $x < z < y$.

Second, we show z is irrational.

Towards a contradiction, assume z is rational. Then $y = 2z - x =$ rational - rational, which must be rational. But we already know y is irrational. A contradiction. So z must be irrational.

So z is the number we are looking for.

7. (0 points). Page 108, Prob 44.

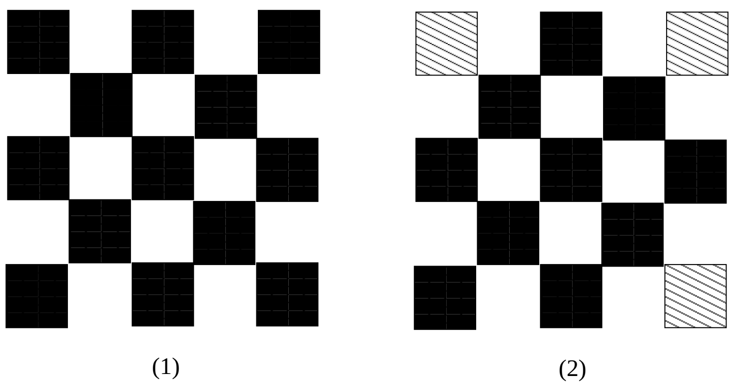

Figure 1: (1) 5×5 checkerboard; (2) after removing three corner.

Sol: We color the squares of 5×5 checkerboard with alternating white and black squares. Without loss of generality, we color the upper left square with black. So, in this coloring, all four corners are black. So there are 13 black squares and 12 white squares. After three corners (all black) are removed, there are 12 white squares and 10 black squares.

Suppose we can use dominoes to tile a 5×5 checkerboard with three corners removed. Since each domino must cover 1 white and 1 black square, we have 11 white squares and 11 black squares covered by dominoes. However, there are 10 black squares and 12 white squares after removing three corner squares. So, it is impossible.

8. (0 points). Prove that there is no positive integer n such that:

$$90 < 2^n \times (n + 1) < 100$$

Sol: Since $f(n) = 2^n \times (n + 1)$ is a strictly increasing function for all $n \geq 0$; it means that $f(n) < f(n + 1)$ for all $n \geq 0$.

Case 1: $n \geq 5$. Then $2^n \times (n + 1) \geq 2^5 \times (5 + 1) = 192 > 100$.

Case 2: $1 \leq n \leq 4$. Then $2^n \times (n + 1) \leq 2^4 \times (4 + 1) = 80 < 100$.

Since the above inequalities hold for both cases, it is true for all positive integer n.