Solution, Assignment #6, CSE 191
Fall, 2014

General Guidelines:

This assignment will NOT be collected, nor graded. However, you should carefully complete it as if it were to be graded. There will be a quiz based on this assignment (with very similar problems) during the week Oct 27 - 31.

The solution will be posted on Friday, Oct. 24.

Note: For problems involving counting, your solution MUST indicate how do you get the answer (such as $C(8,4)\cdot 3!$, $P(5,2)\cdot 2^3$.) This requirement will also apply to the quiz and midterm questions. You do NOT have to give a numerical answer.

1. (0 points). Page 396, Prob 4.

Solution: $12 \times 2 \times 3 = 72$.

2. (0 points). Page 396, Prob 16.

The problem could have two interpretations (with different solutions).

Interpretation 1: Count the number of strings with **exactly one letter** x:

Solution: The number of 4 letter strings with x as the first letter is $25 \times 25 \times 25 = 25^3$. Similarly, the number of 4 letter strings with x as the second letter is also 25^3. The same is true when x is the third and the fourth letter. So the answer is $4 \times 25^3 = 62500$.

Interpretation 2: Count the number of strings with **at least one letter** x:

Solution: The number of 4 letter strings with no restrictions: 26^4.

The number of 4 letter strings that contain **no** x: 25^4.

So the answer is: $26^4 - 25^4 = 66351$

3. (0 points). Page 396, Prob 22, (d), (e), (f)

Solution: Let

A = the set of integers between 1 and 1000 that are divisible by 7.

B = the set of integers between 1 and 1000 that are divisible by 11.

$C = A \cap B$ = the set of integers between 1 and 1000 that are divisible by 77.

Then $|A| = \lceil 1000/7 \rceil = 142$, $|B| = \lceil 1000/11 \rceil = 90$, $|C| = \lceil 1000/77 \rceil = 12$. vspace0.1in

(d) $|A \cup B| = \lceil 1000/7 \rceil + \lceil 1000/11 \rceil - \lceil 1000/77 \rceil = 142 + 90 - 12 = 220$.

(e) $|(A \cup B) - C| = 220 - 12 = 208$.

(f) $999 - |A \cup B| = 1000 - 220 = 779$. (Since it's "positive integers LESS than 1000", the total number involved is 999).

4. (3 points). Page 397, Prob 46, (a), (b), (c)

Solution: (a) There are $P(9,5)$ ways to pick other 5 people (not including the bride). For each of these arrangements, the bride can take 6 different positions. So the answer is $6 \times P(9,5)$.

(b) There are $P(8,4)$ ways to pick other 4 people (not including the bride and the groom). For each of these arrangements, the bride can take 5 different positions. After that, the groom can take 6 different positions. So the answer is $6 \times 5 \times P(8,4)$.

(c) There are $P(8,5)$ ways to pick other 5 people (not including the bride and the groom). For each of these arrangements, the bride can take 6 different positions. So the number of ways that only the bride is in the picture is $6 \times P(8,5)$. Similarly, the number of ways that only the groom is in the picture is also $6 \times P(8,5)$. Thus the answer is $12 \times P(8,5)$.

5. (0 points). Page 405, Prob 4.

Solution: (a) 5.
(b) 13.

6. (0 points). Page 406, Prob 38.

Solution: Let $C_1, C_2, \ldots, C_8$ denote 8 computers and P_1, P_2, P_3, P_4 denote 4 printers.

Connect C_1 to P_1; C_2 to P_2; C_3 to P_3 and C_4 to P_4 (using 4 cables). Then connect each of C_5, C_6, C_7, C_8 to all 4 printers (using 16 cables). So we use total 20 cables. By using the same argument as in Example 9 (page 402), we can show these connections satisfy the requirement.

Now we need to show 20 is the minimum number of cables we need. Suppose that we use at most 19 cables. Then at least one printer, say P_1, is connected to at most 4 computers, say C_1, C_2, C_3, C_4. Then the computers C_5, C_6, C_7, C_8 are connected to P_2, P_3, P_4 only. So the requirement is not satisfied.

7. (0 points). Page 413, Prob 4.

Solution: a) There should be $P(5,3) = 60$ permutations
{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1},
{1, 2, 4}, {1, 4, 2}, {2, 1, 4}, {2, 4, 1}, {4, 1, 2}, {4, 2, 1},
{1, 2, 5}, {1, 5, 2}, {2, 1, 5}, {2, 5, 1}, {5, 1, 2}, {5, 2, 1},
{1, 3, 4}, {1, 4, 3}, {3, 1, 4}, {3, 4, 1}, {4, 1, 3}, {4, 3, 1},
{1, 3, 5}, {1, 5, 3}, {3, 1, 5}, {3, 5, 1}, {5, 1, 3}, {5, 3, 1},
{1, 4, 5}, {1, 5, 4}, {4, 1, 5}, {4, 5, 1}, {5, 1, 4}, {5, 4, 1},
{2, 3, 4}, {2, 4, 3}, {3, 2, 4}, {3, 4, 2}, {4, 2, 3}, {4, 3, 2},
{2, 3, 5}, {2, 5, 3}, {3, 2, 5}, {3, 5, 2}, {5, 2, 3}, {5, 3, 2},
{2, 4, 5}, {2, 5, 4}, {4, 2, 5}, {4, 5, 2}, {5, 2, 4}, {5, 4, 2},
{3, 4, 5}, {3, 5, 4}, {4, 3, 5}, {4, 5, 3}, {5, 3, 4}, {5, 4, 3}
b) There should be $C(5,3) = 10$ combinations
{1,2,3}, {1,2,4},{1,2,5},{1,3,4},{1,3,5},{1,4,5},{2,3,4},{2,3,5},{2,4,5},{3,4,5}

8. (0 points). Page 413, Prob 5, (b) (d)

Solution: b) 720

d) 6720

9. (0 points). Page 413, Prob 6, (b) (c)

Solution: b) 10
c) 70

10. (0 points). Page 413, Prob 12.

Solution: a) $C(12,3) = 220$
b) $C(12,0) + C(12,1) + C(12,2) + C(12,3) = 1 + 12 + 66 + 220 = 299$
c) $\sum_{i=3}^{12} C(12,i) = 2^{12} - C(12,0) - C(12,1) - C(12,2) = 4017$
d) $C(12,6) = 924$

11. (0 points). Page 413, Prob 18.

Solution: (a) 2^8
(b) $C(8,3)$.
(c) $C(8,3)+C(8,4)+C(8,5)+C(8,6)+C(8,7)+(C(8,8)$. Or: $2^8-[C(8,0)+C(8,1)+C(8,2)]$.
(d) $C(8,4)$.

12. (0 points). Page 414, Prob 22, (d) (e) (f).

Solution: (d) In this case, AB acts as a single symbol, DE acts as a single symbol, GH acts as a single symbol. So there are actually 5 symbols that can be arranged independently. Thus the answer is 5!.

(e) In this case, CABED acts as a single symbol. So we have 4 independent symbols. The answer is 4!.

(f) No such permutation exists. Answer is 0.

13. (0 points). Page 414, Prob 26.

Solution: (a) $C(13,10)$.
(b) $P(13,10)$
(c) 1 woman team: $C(3,1) \times C(10,9)$; 2 women team: $C(3,2) \times C(10,8)$; 3 women team: $C(3,3) \times C(10,7)$.
So the answer is: $C(3,1) \times C(10,9) + C(3,2) \times C(10,8) + C(3,3) \times C(10,7)$.

14. (0 points). Page 421, Prob 8.

Solution:
$$(3x + 2y)^{17} = \ldots + C(17,8) \cdot (3x)^8 \cdot (2y)^9 + \ldots$$
So the coefficient of x^8y^9 is: $3^8 \cdot 2^9 \cdot C(17,8)$.

15. (0 points). Page 421, Prob 12.

Solution: 1 11 55 165 330 462 462 330 165 55 11 1

16. (0 points). Page 422, Prob 22.

Solution: Prove

$$C(n,r)\cdot C(r,k) = C(n,k)\cdot C(n-k,r-k) \tag{1}$$

(a) Combinatorial argument:

Consider this counter problem: We have a set S with n elements. Count the number of ways to select a subset A with r elements, then the number of ways to select a sub-subset B with k elements within each r-element subset A.

The number of choices for A is $C(n,r)$. The number of choices for B within each A is $C(r,k)$. So the total number of choices for the pairs (A,B) is $C(n,r)\cdot C(r,k)$, which is the LHS of (1).

Now we count the number of such (A,B) pairs in a different way.

First we directly select a sub-subset B with k elements from S. The number of such choices is $C(n,k)$. For each such B, we "enlarge it" to A by adding $r-k$ elements from the other $n-k$ elements in S. So there are $C(n-k,r-k)$ different ways to enlarge B to an r-element set A. Thus, by this counting method, the number of (A,B) pairs is $C(n,k)\cdot C(n-k,r-k)$, which is the RHS of (1).

Because we are counting the same set of abjects (the (A,B) pairs), the result of the two counting methods must be the same. Thus the equation (1) is true.

(b) LHS $= C(n,r)\cdot C(r,k) = \frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!} = \frac{n!}{(n-r)!k!(r-k)!}$.
RHS $= C(n,k)\cdot C(n-k,r-k) = \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-k-r+k)!} = \frac{n!}{(n-r)!k!(r-k)!}$.
Thus LHS = RHS.

17. (0 points). Page 422, Prob 33.

Solution:

(a) Each path from the point$(0,0)$ to the point (m,n) consists of $m+n$ line segments (each with length 1). It must have m horizontal line segments, and n vertical line segments. We use a $m+n$ bit binary string to describe a path: 0 represents a horizontal line segment, and 1 represents a vertical line segment. (For example: The path in the first figure on page 422 is represented by: 01100010. The path in the second figure on page 422 is represented by: 10000011.)

(b) Thus the number of such paths is the same as the number of $m+n$ bit binary strings containing n 1's. This number is $C(n+m,n)$.

18. (4 points) There are 8 students trying to solve 8 problems. Each problem is solved by at least 5 students. Prove that there exist two students such that every problem is solved by at least one of them.

The easiest way to prove it is perhaps by using Pigeonhole Principle. But you can prove it by any other method that works.

Sol 1: Let $A = \{S_1, S_2, \ldots, S_8\}$ be the eight students and $P = \{P_1, P_2, \ldots, P_8\}$ be the eight problems.

Since each problem is solved at least 5 times, the eight problems are collectively solved at least 40 times (these are the pigeons) by 8 students (these are the pigeonholes). By Pigeonhole Principle, at lease one student solved at least $\lceil 40/8\rceil = 5$ problems. (Another way to think about it: If every student solved at most 4 problems, then they collectively solved the problems at most $4\times 8 = 32$ times. It would be impossible that each problem is solved at least 5 students.)

Without loss of generality, we assume S_1 solved at least 5 problems.

Case 1: S_1 solved all 8 problems. Then S_1 and any another student satisfy the conclusion of the claim.

Case 2: S_1 solved 7 problems $P_1 \dots P_7$. Then S_1 and any student who solved P_8 satisfy the conclusion of the claim.

Case 3: S_1 solved 6 problems $P_1 \dots P_6$, but not P_7 and P_8. The remaining two problems (P_7 and P_8) are collectively solved at least $5 \times 2 = 10$ times by the other seven students. So at least one other student must have solved these two problems at least $\lceil 10/7 \rceil = 2$ times. In other words, this student solved both P_7 and P_8. So S_1 and this student satisfy the conclusion of the claim.

Case 4: S_1 solved 5 problems $P_1 \dots P_5$, but not P_6, P_7, P_8. The remaining three problems (P_6, P_7, P_8) are collectively solved at least $5 \times 3 = 15$ times by the other seven students. So at least one other student must have solved these three problems at least $\lceil 15/7 \rceil = 3$ times. In other words, this student solved all three problems P_6, P_7, P_8. So S_1 and this student satisfy the conclusion of the claim.

Solution 2. (This solution is shorter, but harder to follow).

There are $C(8,2) = 28$ pairs of students. Let's take 28 boxes, and label each box by a pair of students.

Consider any problem P_i. P_i is solved by at least 5 students. So P_i is NOT solved by at most 3 students. So P_i is NOT solved by at most $C(3,2) = 3$ pairs of students. If P_i is not solved by a pair of students, put a card (with P_i on it) into the box labeled by that student pair.

Thus there are at most 3 cards for each problem P_i. Hence there are at most $8 \times 3 = 24$ cards for all 8 problems. When you put 24 cards into 28 boxes, at least one box must be empty. Say this empty box is labeled by Student i and Student j. This means that no problem is NOT solved by either Student i or Student j. In other words, every problem is solved either by Student i or Student j. This is what we want to prove.