Solution #9, CSE 191

Fall, 2014

Solution

1. (0 points).

Sol: Let $P(n)$ denote the proposition: " $\sum_{i=1}^{n} i^3 = \frac{n^2\cdot(n+1)^2}{4}$".

BASIS STEP: $P(1)$ is true, because $1^3 = \frac{1^2\cdot(1+1)^2}{4}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $\sum_{i=1}^{k} i^3 = \frac{k^2\cdot(k+1)^2}{4}$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{(by the meaning of } \textstyle\sum) \\
&= \frac{k^2\cdot(k+1)^2}{4} + (k+1)^3 && \text{(by induction hypothesis.)} \\
&= (k+1)^2\cdot[\frac{k^2}{4} + (k+1)] && \text{(the rest are by algebra.)} \\
&= (k+1)^2\cdot[\frac{k^2+4k+4}{4}] \\
&= \frac{(k+1)^2\cdot(k+2)^2}{4}
\end{aligned}
$$

This establishes the inductive step of the proof.

2. (0 points). Page 330, Problem 10 (a), (b).

Sol:

(a):

$$
\begin{aligned}
\frac{1}{1\cdot 2} &= \frac{1}{2} \\
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} &= \frac{2}{3} \\
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} &= \frac{3}{4}
\end{aligned}
$$

We guess $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(b): Let $P(n)$ denote the proposition: "$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$".

BASIS STEP: $P(1)$ is true, because $\frac{1}{1\cdot 2} = \frac{1}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$.

$$
\begin{aligned}
&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} && \text{(by the meaning of } \textstyle\sum) \\
=&\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} && \text{(by induction hypothesis.)} \\
=&\frac{1}{k+1}\cdot(k+\frac{1}{k+2}) && \text{(the rest are by algebra.)} \\
=&\frac{1}{k+1}\cdot(\frac{k^2+2k+1}{k+2}) \\
=&\frac{1}{k+1}\cdot\frac{(k+1)^2}{k+2} \\
=&\frac{k+1}{k+2}
\end{aligned}
$$

This establishes the inductive step of the proof.

3. (0 points). Page 330, Problem 14.
Sol:
Let $P(n)$ denote the proposition: " $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1}+2$".

BASIS STEP: $P(1)$ is true, because $1\cdot 2^1 = (1-1)\cdot 2^{1+1}+2$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $\sum_{i=1}^{k} i2^i = (k-1)2^{k+1}+2$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} i2^i &= \sum_{i=1}^{k} i2^i + (k+1)\cdot 2^{k+1} && \text{(by the meaning of } \textstyle\sum) \\
&= (k-1)\cdot 2^{k+1}+2+(k+1)\cdot 2^{k+1} && \text{(by induction hypothesis.)} \\
&= 2^{k+1}\cdot(k-1+k+1)+2 && \text{(the rest are by algebra.)} \\
&= 2^{k+1}\cdot 2\cdot k+2 \\
&= k\cdot 2^{k+2}+2
\end{aligned}
$$

This establishes the inductive step of the proof.

4. (0 points). Page 330, Problem 18.
Sol:

(a): $2! < 2^2$

(b): $2! = 2\cdot 1 = 2$;
$2^2 = 4$;
So $2! < 2^2$.

(c): The inductive hypothesis is the statement that $P(k)$ is true, that is, $k! < k^k$.

(d): $(k+1)! < (k+1)^{k+1}$.

(e):

$$\begin{aligned} LHS = (k+1)! = k! \cdot (k+1) & \quad \text{(by the definition of } (k+1)!) \\ < k^k \cdot (k+1) & \quad \text{(by induction hypothesis.)} \\ < (k+1)^k \cdot (k+1) & \quad \text{(because } k^k < (k+1)^k) \\ = (k+1)^{k+1} & \end{aligned}$$

(f): Toward a contradiction, assume that there is at least one positive integer greater than 1 for which $P(n)$ is false. Let S be the set of positive integers greater than 1 for which $P(n)$. Then $S \neq \emptyset$. Let m be the smallest integer in S. We know that m cannot be 2, because $P(2)$ is true. Because m is positive and greater than 2, $m-1$ is a positive integer greater than 1. Furthermore, because $m-1$ is less than m, it is not in S, so $P(m-1)$ must be true. Because the conditional statement $P(m-1) \to P(m)$ is also true, it must be the case that $P(m)$ is true. This contradicts the choice of m. Hence, $P(n)$ must be true for every positive integer n.

5. (0 points). Page 330, Problem 20.
Sol: Let $P(n)$ denote the proposition: "$3^n < n!$".

BASIS STEP: $P(7)$ is true, because $3^7 < 7!$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $3^k < k!$.

$$\begin{aligned} 3^{k+1} &= 3^k \times 3 & \text{(by the meaning of } 3^{k+1}.) \\ &< k! \times 3 & \text{(by induction hypothesis.)} \\ &< k! \times (k+1) & \text{(because } 3 < k+1.) \\ &= (k+1)! \end{aligned}$$

This establishes the inductive step of the proof.

6. (3 points). Page 330, Problem 32.
Sol: Let $P(n)$ denote the proposition: "3 divides $n^3 + 2n$".

BASIS STEP: $P(1)$ is true, because 3 divides $1^3 + 2 \times 1 = 3$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, 3 divides $k^3 + 2k$. So $k^3 + 2k = 3h$ for some positive integer h.

$$\begin{aligned} (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 & \text{(by algebra.)} \\ &= (k^3 + 2k) + 3(k^2 + k + 1) & \text{(by algebra.)} \\ &= 3h + 3(k^2 + k + 1) & \text{(by induction hypothesis.)} \\ &= 3(h + k^2 + k + 1) & \text{(by algebra.)} \end{aligned}$$

So, 3 divides $(k+1)^3 + 2(k+1)$. This establishes the inductive step of the proof.

7 (0 points). Page 331, Problem 40.

Sol: Let $P(n)$ denote the proposition: "$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$".

BASIS STEP: $P(1)$ is true, because $A_1 \cap B = A_1 \cap B$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $(A_1 \cup A_2 \cup \cdots \cup A_k) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)$.

$$\begin{aligned} RHS =& [(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)] \cup (A_{k+1} \cap B) && \text{(by the meaning of set operations.)} \\ &= ((A_1 \cup A_2 \cup \cdots \cup A_k) \cap B) \cup (A_{k+1} \cap B) && \text{(by induction hypothesis.)} \\ &= (A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}) \cap B && \text{(by distributive rule for set operations.)} \\ &= LHS \end{aligned}$$

This establishes the inductive step of the proof.

8. (0 points). Page 331, Problem 51.

Sol: The mistake is in applying the inductive hypothesis to look at $\max(x-1, y-1)$, because even though x and y are positive integers, $x-1$ and $y-1$ need not be (one or both could be 0). In fact, that is what happens if we let $x = 1$ and $y = 2$ when $k = 1$.